The Complete Information First–Price Auction with Finite Bidding Grid*

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Abstract

Despite the popularity of auction theoretical thinking, it appears that no one has presented an elementary equilibrium analysis of the complete information first-price sealed-bid auction mechanism when the bidding space has a finite grid. This paper aims to remedy that omission. We show that there always exists a "high price equilibrium" which can be considered "the intuitive solution". However,–contrary to conventional wisdom–in many instances this equilibrium is inefficient. There might also be inefficient "low price equilibria" which do not automatically go away as the smallest monetary unit diminishes. Moreover, when the bidding space is very restrictive the revenue obtained in these "low price equilibria" might be very low.

Keywords: First-price auctions, undominated Nash equilibria.

JEL Classification Numbers:

C72 (Noncooperative Games),

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1. Introduction

It is well known among auction theorists that the first-price sealed-bid auction mechanism under complete information does not possess a pure strategy Nash equilibrium. For instance, Moldovanu and Sela (2003, footnote 12) write that "asymmetric Bertrand games (and first-price auctions) have no equilibria in pure strategies here, but introducing a smallest money unit immediately yields the intuitive solution." Conventional wisdom holds that in this intuitive solution the bidder with the highest valuation wins the auction, which is, thus, efficient.[1] However, despite the popularity of auction theoretical thinking, it appears that no one has presented an elementary equilibrium analysis of the first-price sealed-bid auction mechanism under these conditions. This note aims to remedy that omission.

There are several reasons why an analysis under complete information merits a closer look. First, these settings are often considered to be a useful starting point of the analysis before moving to incomplete information models or to be a useful benchmark case.[2] Second, auction-theoretic ways of thinking have been successfully applied to the

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[1] To be fully precise, it can be shown that there is no undominated pure strategy Nash equilibrium for the first-price auction and the only case in which a pure strategy Nash equilibrium exists is when two bidder with the highest valuation have the same valuation. This is due to the discontinuity of the payoff functions when two bidders tie at the highest bid and parallels a well-known result in the related game of Bertrand competition (see our more extensive working paper version Alcalde and Dahm (2008)). A first-price auction and a Bertrand oligopoly market in which firms produce a homogeneous good at constant marginal costs are in general not equivalent, as in the latter the winning firm's demand decreases in the price. This corresponds to an auction in which the winning bidder's valuation increases in her bid.

[2] For the former see e.g. Baye et al. (1993) and (1996), Benoit and Krishna (2001), Bernheim and

analysis of broader economic questions (see Klemperer (2003)) and for some applications the benchmark of complete information has been argued to more appropriate than an incomplete information setting.[3]

Although, there are alternative ways to restore existence of equilibrium–like looking for mixed-strategy equilibria–, our approach of a bidding space with a finite grid is important. First, as the above quote shows it is a very natural procedure. Second, in experimental settings there is also a smallest monetary unit. Third, this model is often viewed as a better description of reality.[4] Notice also that in applications the relevant bidding space might be very irregular and the required increments of bids might be large. For instance, when several jurisdictions compete for the location of a new factory, they either commit to constructing a regional airport, an entrance to a motorway or to expand the harbor or they do not commit. They cannot provide just a

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Whinston (1986) or Krishna and Tranaes (2002)). For the latter see e.g. Anton and Yao (1989), Grimm et al. (2003), Elmaghraby and Oren (1999), or Moldovanu and Sela (2003)).

[3]For instance, Moldovanu and Sela (2003) use the first-price auction mechanism to model patent licensing. In their view the benchmark of complete information is appropriate for mature industries, like the steel industry. For this industry they report that competitors "know each other well, and engineers often visit competitors' plants". However, they argue that emerging or very dynamic and secretive industries, like the petrochemical industry, are better captured by an incomplete information model.

[4]Simon and Zame (1990, p. 863) state this view as follows. "Games with infinitely many strategies are sometimes viewed as proxies for games with a large finite number of strategies. From this point of view it is the equilibria ... of the finite games which are of real interest; equilibria of the infinite games are merely convenient approximations." Rapoport and Amaldoss (2004, p. 587) write "the assumption of a discrete strategy space is appropriate as firms typically consider their expenditures in discrete (e.g., thousands or millions of dollars) rather than continuous units. Indeed, continuous strategy spaces are mostly introduced to achieve tractability, not to provide a more adequate description of reality". Other auction models using the assumption of a finite grid are Shubik (1971), O'Neill (1986), Chwe (1989) or Rapoport and Amaldoss (2004).

fraction of an airport.[5]

The present note offers an elementary analysis of pure strategy undominated Nash equilibria assuming fairly general tie-breaking rules and (possibly irregular) finite grids on bidding spaces. We show that there always exists the intuitive "high price equilibrium" which contrary to conventional wisdom might be inefficient. There might also be inefficient "low price equilibria" which do not automatically go away as the smallest monetary unit diminishes. Moreover, when the bidding space is very restrictive the revenue obtained in these "low price equilibria" might be very low.

2. The Model and Notation

Consider the seller of an (indivisible) object (indexed by 0) and a set of potential buyers $\mathcal{B} = \{B_1, \ldots, B_i, \ldots, B_n\}$. Each agent has a valuation v_i for the object. There are at least two buyers, i.e. $n \geq 2$, and agents' valuations are increasingly ordered, i.e. $v_i \leq v_j$ for all $0 \leq i \leq j$. All agents' valuations $v = (v_0, v_1, \ldots, v_i, \ldots, v_n)$ are commonly known by all the buyers, and this is public information. As in Bernheim and Whinston (1986) or Anton and Yao (1989) the seller only has information about her own valuation of the object.

The finite bidding grid is formalized as follows. Let the (fixed) set of prices that buyers can propose be given by $A = \{a_0, a_1, ..., a_k, ..., a_K\}$, where $a_k \in \mathbb{R}^n_+$ and $a_{k+1} \geq a_k$

[5] See Menezes (2003) for an auction theoretic analysis of such a situation.

for all $k = 0, 1, ..., K-1$. For each such k, define $\delta_{a_k} = a_{k+1} - a_k$. We say that the bidding space has a finite grid if there exists $\delta > 0$ such that $\delta_{a_k} \geq \delta$ for all δ_{a_k}. If all δ_{a_k} are equal, we say that the bidding space has a constant grid of (at least) size δ.

We formalize now the first-price auction mechanism analyzed in the present paper. Loosely speaking, the object is assigned to the buyer with highest bid, and she pays her bid. However, when two or more buyers propose the same bid there is a function π establishing a probabilistic allocation rule in order to assign the object. This fixed monotonic (probabilistic) measure function

$$\pi : 2^{\mathcal{B}} \to \mathbb{R}^n$$

satisfies:

(a) for all $S \subseteq \mathcal{B}$, $\sum_{i=1}^{n} \pi_i(S) = 1$,

(b) for all $S \subseteq \mathcal{B}$ and $i \in \mathcal{B} \backslash S$, $\pi_i(S) = 0$,

(c) for all $S \subseteq \mathcal{B}$ and $i \in S$, $\pi_i(S) > 0$; and

(d) for all $S \subseteq S' \subseteq \mathcal{B}$, and $i \in S$, $\pi_i(S) \geq \pi_i(S')$.

For A and π given, the first-price auction mechanism proceeds as follows.[6] Each

[6]Notice that the following can be motivated by a two-stage game in which in the first stage each buyer simultaneously bids p_i and in the second stage the seller establishes her reservation price. Given that it is a dominant strategy for the seller to behave truthfully, at any subgame perfect equilibrium the object will be sold whenever some buyer sets a price higher than v_0. Therefore, we concentrate on the analysis of buyers' decisions at the first stage.

buyer simultaneously sets the price $p_i \in A$ $(i = 1, \ldots, n)$ that she is willing to pay for the object if it is assigned to her. This defines a vector $p = (p_1, \ldots, p_n)$.

(1) If $p_i < v_0$ for all $i = 1, \ldots, n$, the object is unassigned, i.e. the seller keeps it.

(2) Otherwise, denote by $S(p) = \{B_i \in \mathcal{B} : p_i \geq p_j \text{ for all } B_j \in \mathcal{B}\}$ the set of buyers proposing the highest bid. Then the object is assigned with probability $\pi_i(S(p))$ to buyer B_i who pays p_i with this probability.

We analyze undominated Nash equilibria (in pure strategies) in the bidding game. Note that for each buyer B_i, a strategy $\hat{p}_i$ is undominated if, and only if, $0 \leq \hat{p}_i < v_i$.[7] We denote by $w_i \in A$ agent B_i's largest (undominated) bid strictly smaller than v_i. For simplicity we also denote $\delta_{w_{n-1}} = \Gamma$.

3. Analysis of the First-Price Auction Mechanism

It turns out that if a strategy profile is an equilibrium, then it belongs to the following class of strategy profiles.

Definition 3.1. *Given $a \in A$, $a \leq \min\{w_{n-1} + \Gamma, w_n\}$, we denote by $\mathcal{P}(a)$ the set of strategy profiles $\hat{p}$ is such that:*

1. *Buyer B_n chooses $\hat{p}_n = a$.*

[7] This rules out Nash equilibria of the following kind. Suppose that only integer bids are feasible, there are two buyers with $v_1 = 1$ and $v_2 = 10$, and the seller's reserve price is zero. For instance, $p_1^* = 7$ and $p_2^* = 8$ constitute a Nash equilibrium (as long as π is not too biased towards B_2).

2. *There exists* $B_j \in \mathcal{B} \setminus \{B_n\}$ *such that* $w_j = w_{n-1}$ *bidding* $\hat{p}_j = \min\{a, w_{n-1}\}$.

3. *All other bidders* $B_i \in \mathcal{B} \setminus \{B_j, B_n\}$ *choose* $\hat{p}_i \leq \min\{w_i, a\}$.

Notice that $a \in A$ just indicates the winning bid.[8] Given a strategy profile $\hat{p} \in \mathcal{P}(a)$, we indicate the buyers bidding at least $b \in A$ by $W(b) = \{B_i \in \mathcal{B} \text{ s.t. } \hat{p}_i \geq b\}$. To simplify notation we will omit a and b using δ and W instead, whenever this notation is clear from the context.

Given a winning bid a and a strategy profile $\hat{p} \in \mathcal{P}(a)$, it might pay to raise or lower an individual bid. Define the two threshold values

$$
\begin{aligned}
\alpha &= [1 - \pi_n (W(w_{n-1}))] [v_n - w_{n-1}] \qquad \text{and} \\
\beta &= \left.\begin{aligned} &\max \quad [1 - \pi_j (W(a))] [v_j - a] \\ &s.t \qquad B_j \in W(a). \end{aligned}\right\}
\end{aligned}
$$

Notice that the definition of β might not be determined by B_n when the probabilistic measure function π is strongly biased in favor of this buyer.[9]

We are now in a position to characterize undominated Nash equilibria when the bidding space has a finite grid. There are three cases to be distinguished. Case (1) and case (2.2) formalize the conventional wisdom that the strongest bidder just outbids the

[8] We implicitly assume in what follows that $v_0 \leq a$.

[9] The exact threshold for B_n not to determine β is that there exists $B_i \in W \backslash B_n$ such that $\pi_n > 1 - (1 - \pi_i)(v_i - a)/(v_n - a)$.

others or ties with an equally strong bidder at their common valuation. However, case (2.1) shows that even when valuations are different it might not pay to outbid others because the required increase of the bid may be too large. Case (3) establishes that this intuition might even apply to much lower bids.

Theorem 3.2. *A profile of strategies p^* is an undominated Nash equilibrium for the first-price auction if, and only if, $p^* \in \mathcal{P}(a)$ for some $a \in A$; and one of the following is true:*

(1) (High price equilibrium, unique winner) $a = w_{n-1} + \Gamma$, $w_{n-1} < w_n$ and $\Gamma \leq \alpha$.

(2) (High price equilibrium, tie) $a = w_{n-1}$ and either

(2.1) $w_{n-1} < w_n$ and $\Gamma \geq \alpha$ or

(2.2) $w_{n-1} = w_n$.

(3) (Low price equilibrium, tie) $a < w_{n-1}$ and $\delta_a \geq \beta$.

Proof. (I) We show first that p^* is an undominated Nash equilibrium for the first-price auction whenever (1), (2) or (3) are true. Note that, since $p_i^* \leq w_i$ for all $B_i \in \mathcal{B}$, no buyer employs a dominated strategy. We show now that p^* is a Nash equilibrium. Let us observe that the expected utility of buyers in $\mathcal{B} \setminus W$ is zero. Moreover, given agents' bids, no buyer in $\mathcal{B} \setminus W$ can obtain a positive (expected) utility.

Suppose (1) holds. The fact that $p^* \in \mathcal{P}(a)$ implies that B_n wins, so

$$U_n(p^*) = v_n - a \geq 0, \text{ with } a = w_{n-1} + \Gamma.$$

Assume B_n changes p_n^* to $\tilde{p}_n$. Given that she cannot gain from raising her bid, suppose $\tilde{p}_n \leq w_{n-1}$. Notice that there exists $B_j \neq B_n$ bidding $\hat{p}_j(a) = w_{n-1}$. We have

$$\begin{aligned} U_n\left(\tilde{p}_n, p_{-n}^*\right) &\leq \pi_n\left(W(w_{n-1})\right)\left[v_n - w_{n-1}\right] = v_n - w_{n-1} - \alpha \leq \\ &\leq v_n - w_{n-1} - \Gamma = v_n - a = U_n(p^*). \end{aligned}$$

And, thus, p_n^* is the best decision for agent B_n, given the others' bids.

Suppose (2) or (3) holds. For buyer $B_j \in W$, we have that her expected utility is

$$U_j(p^*) = \pi_j(W)\left[v_j - p_j^*\right] = \pi_j(W)\left[v_j - a\right] > 0.$$

Assume B_j changes her strategy, by setting $\tilde{p}_j$. If she lowers her bid, her expected utility will be zero, since W is not a singleton. Thus, suppose $\tilde{p}_j > a$, and note that B_j will get the object with probability one. Notice that in case (2.2) $U_j\left(\tilde{p}_j, p_{-j}^*\right) < 0$. Consider case (3). Observe that

$$\begin{aligned} U_j\left(\tilde{p}_j, p^*_{-j}\right) &= v_j - \tilde{p}_j \leq v_j - a - \delta_a \leq v_j - a - \beta \leq \\ &\leq v_j - a - \left[1 - \pi_j\left(W\right)\right]\left[v_j - a\right] = U_j\left(p^*\right). \end{aligned}$$

Again, p^*_j is the best decision for agent B_j, given the others' bids. The argument for case (2.1) is similar replacing B_j, δ_a and β by B_n, Γ and α respectively.

(II) We show now the converse. Suppose there is a Nash equilibrium p^* in which all agents employ undominated strategies. Denote the highest bid by

$$a' = \max_{B_i \in \mathcal{B}} p^*_i.$$

Notice that if $a' > \min\{w_{n-1} + \Gamma, w_n\}$, then a' is either dominated or $p^*_n = a'$. In the latter case B_n can improve by lowering her bid. Hence, suppose $a' \leq \min\{w_{n-1}+\Gamma, w_n\}$. Notice that $p^*_n = a'$ must hold because otherwise B_n can improve by making this bid. Suppose $a' = w_{n-1} + \Gamma$ and that there does not exist $B_j \in \mathcal{B} \setminus \{B_n\}$ such that $w_j = w_{n-1}$ bidding $p^*_j = w_{n-1}$. Given that for bidders with lower valuations $p^*_i = w_{n-1}$ is dominated, B_n could improve by lowering her bid. Assume $a' \leq w_{n-1}$ and that there exists $B_j \in \mathcal{B} \setminus \{B_n\}$ such that $w_j \geq a'$ bidding $p^*_j < a'$. In this case B_j can improve by changing her bid to $\tilde{p}_j = a'$ because

$$U_j(p^*) = 0 < \pi_j\left(W(a') \cup B_j\right)\left[v_j - a'\right] = U_j\left(\tilde{p}_j, p^*_{-j}\right)$$

This proves that $p^* \in \mathcal{P}(a')$. From part (I) it is clear p^* cannot be a undominated Nash equilibrium when the conditions in case (1), (2) or (3) are not fulfilled. ■

Observe that case (1) and case (2) of Theorem 3.2 imply the following.

Corollary 3.3. *There exists a 'high price' undominated Nash equilibrium in the first-price auction. In this equilibrium the winning bid a fulfills $a \in \{w_{n-1}, w_{n-1} + \Gamma\}$.*

However, in addition to this equilibrium there might be further low price equilibria as specified in case (3) of Theorem 3.2. The intuition for the existence of the low price equilibria is the following. A bidder can prevent a tie by outbidding the competitors by the minimal increase. However, when the grid is restrictive the required increase is large and does not pay. So a natural question to ask is, How small must a smallest monetary unit be in order to make sure that low price equilibria do not exist? Assume that the tie breaking rule assigns the object with equal probability and that there are two bidders.[10]

[10] Notice that because of the monotonicity of the tie breaking rule further bidders increase the incentives to deviate from a low price equilibrium. For completeness we mention that case (1) of the next Corollary assumes that $\Gamma \neq \alpha$.

Corollary 3.4. *Assume that there are two bidders who get the object with equal probability in case of a tie and that the bidding space has a constant grid of size δ. For any $\delta > 0$ the following is true:*

(1) If $w_{n-1} < w_n$, the high price equilibrium is unique.

(2) If $w_{n-1} = w_n$, in addition to the high price equilibrium, there exists a low price equilibrium with strategy profile[11] *$\hat{p}(a)$ where $a = w_{n-1} - \delta$.*

Proof. Suppose $w_{n-1} < w_n$. The fact that $v_n > w_n \geq w_{n-1} + \delta$ implies that $\delta < v_n - w_{n-1}$. We show first that the profile $\hat{p}(a)$ with $a = w_{n-1} - \delta$ is not an equilibrium. For this $\delta < \beta$ must hold. Since $\beta = \frac{1}{2}(v_n - w_{n-1} + \delta)$, $\delta < \frac{1}{2}(v_n - w_{n-1} + \delta)$ must hold. Simplifying yields $\delta < v_n - w_{n-1}$, which we already have shown to be true. Notice that no profile $\hat{p}(a')$ with $a' < w_{n-1} - \delta$ can be an equilibrium because $\beta(a) = \frac{1}{2}(v_n - a) < \frac{1}{2}(v_n - a') = \beta(a')$.

Suppose $w_{n-1} = w_n$. Notice that $v_n - w_n \leq \delta$. This implies that $\beta = \frac{1}{2}(v_n - w_{n-1} + \delta) \leq \delta$. ∎

This result implies, on one hand, that whenever $v_{n-1} = v_n$, there are at least two equilibria. Note also that for the second statement to hold it is not needed that the grid is constant. It is sufficient that the discrete jump immediately before w_{n-1} is larger

[11] Note that, for the two-bidder case, for any $a \in A$ such that $a \leq \min\{w_{n-1} + \Gamma, w_n\}$, $\mathcal{P}(a)$ is a singleton. Therefore, as we do throughout this corollary and its proof, we can denote by $\hat{p}(a)$ such an element.

or equal than the one after w_{n-1}. On the other hand, when valuations are different decreasing the size of the smallest monetary unit assures that up from a certain point $w_{n-1} > w_n$ holds and the high price equilibrium is unique. We give now an example in which increasing the restrictiveness of the bidding space creates multiple equilibria.

Example 3.5. *There are two bidders with valuations* $v_1 = 90$ *and* $v_2 = 100$. *The reservation price of the seller is zero. In the case that both bidders submit the same bid, they obtain the object with equal probability. Suppose first that the bidding space coincides with the set of odd integers. In this case Theorem 3.2(1) and Corollary 3.4(1) imply that* $p^* = (89, 91)$ *is the unique undominated Nash equilibrium. However, if bidding space is* $A = \{1, 51, 76, 89, 91, 99, 106, ...\}$, *then apart from* p^* *there are three additional equilibria, namely,* $p^{*\prime} = (1, 1)$, $p^{*\prime\prime} = (51, 51)$ *and* $p^{*\prime\prime\prime} = (76, 76)$. *Notice that, although* A *is restrictive, it still leaves the bidders a fairly rich set of options.*

4. Concluding Remarks

This note has presented an elementary equilibrium analysis of the complete information first-price sealed-bid auction mechanism when the bidding space has a finite grid.

We have shown that in addition to the 'intuitive' high price equilibrium, there might be low price equilibria. Example 3.5 has shown that whether a tie at a low price is an equilibrium depends on the local properties of the bidding space at this price. Moreover, when $v_{n-1} = v_n$ there are in many situations at least two equilibria.[12]

Low price equilibria may generate considerably lower revenues than high price equilibria. In this sense there is 'collusion'. But, given that bidding strategies constitute an equilibrium, they are also 'self-enforcing'. This contrasts with the conventional wisdom that "unlike in a second-price auction, the cartel agreement in a first-price auction is *not* self-enforcing and, hence, is somewhat fragile" (Krishna (2002), pg. 160).

In many instances the equilibrium–including the 'intuitive' high price equilibrium–is inefficient because the object is not awarded to the bidder with the highest valuation with positive probability.[13]

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[12] Notice also that the definition of the equilibrium strategies implies that the undominated Nash equilibrium is not unique, as a bidder with a low valuation may submit any undominated bid. Moreover, when $w_{n-1} = w_n$ and $\Gamma = \alpha$ there are two high price equilibria.

[13] In case (2.1) of Theorem 3.2 the equilibrium is inefficient. In addition, the same is true when case (2.2) applies but $v_{n-1} < v_n$ holds or when in case and (3) there exists $B_i \in W$ with $v_i < v_n$.

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