

## 6. The Solow-Swan growth model

### 6.1. Description

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The economy evolves in discrete time. Time is measured in periods, denoted by  $t$ , and indexed by integers:  $t \in \{1, 2, 3, \dots\}$ .

There is only one good in each period. The good is produced. The amount of good in  $t$  is  $Y(t)$ . Its price is normalized to 1 in every  $t$ .

There are two types of agents: consumers (individuals or families) and firms.

Consumers are globally characterized by a constant saving rate  $s$ : in total, each  $t$ , consumers save a fixed proportion  $s$  of  $Y(t)$ .

Consumers supply labour inelastically in a competitive market in exchange for a wage rate  $\omega(t)$ . Labour is identified with employment. The total supply of labour (total employment) in period  $t$  is  $L(t)$  and is assumed to grow at a constant rate  $n$ .

Firms are represented by a profit-maximizing aggregate firm that produces the good using an aggregate production function  $Y(t) = F(A(t), K(t), L(t))$  exhibiting constant returns to scale [alternative interpretation: all firms have access to the same (free) production technology].

$K(t)$  is the capital stock: good used to produce more good. The good can be consumed by consumers or used in production by firms.

$A(t)$  represents the state of technology in period  $t$ . It is defined by a number but it has no natural units. It is rather an index for a family of production functions that depend on just  $K$  and  $L$ .  $A(t)$  captures everything affecting the way of using  $K$  and  $L$  efficiently.

The production function is assumed to have the following properties.

- (i) It takes non-negative values:  $F \geq 0$ .
- (ii) It is twice differentiable with respect to  $K$  and  $L$ .
- (iii) Positive marginal productivity of  $K$  and  $L$ :  $F_K := \frac{\partial F}{\partial K} > 0$  and  $F_L := \frac{\partial F}{\partial L} > 0$ .
- (iv) Decreasing marginal productivity of  $K$  and  $L$ :  $F_{KK} := \frac{\partial^2 F}{\partial K^2} < 0$  and  $F_{LL} := \frac{\partial^2 F}{\partial L^2} < 0$ .
- (v) It exhibits constant returns to scale (is homogeneous of degree 1) in  $K$  and  $L$  [ $F$  is homogeneous of degree  $h$  in  $K$  and  $L$  if, for all  $\lambda > 0$  and  $A$ ,  $F(A, \lambda K, \lambda L) = \lambda^h F(A, K, L)$ ].

The above assumptions make  $F$  concave. They also imply that  $F(A, K, 0) = F(A, 0, L) = 0$ .

**Euler's homogeneous function theorem (Euler's adding-up theorem).** If  $F(A, K, L)$  is twice differentiable with respect to  $K$  and  $L$ , and homogeneous of degree  $h$  in  $K$  and  $L$ , then:

(i) for all  $(A, K, L)$ ,  $h \cdot F(A, K, L) = F_K \cdot K + F_L \cdot L$ ;

(ii)  $F_K$  and  $F_L$  are homogeneous of degree  $h - 1$  in  $K$  and  $L$ .

If  $h = 1$ , then  $F(A, K, L) = F_K \cdot K + F_L \cdot L$  means that output  $F$  is exhausted if factors of production are paid according to their marginal productivities.

All markets (for output  $Y$ , capital  $K$ , labour  $L$ ) are competitive. The price of  $K$  and  $L$  in period  $t$  are denoted, respectively, by  $\omega(t)$  and  $\sigma(t)$ .

The capital stock  $K$  depreciates at a constant rate  $\delta > 0$ : 1 unit of  $K$  in period  $t$ , becomes  $1 - \delta$  units in period  $t + 1$ .

The (real) interest rate is  $r(t) = \sigma(t) - \delta$ . If one unit of the good is used as capital at the start of time  $t$ , then, at the end of  $t$ , its marginal productivity (equal to  $\sigma(t)$ , see below) is obtained, but a fraction  $\delta$  of the capital is lost.

Firms maximize profits and can be represented by a unique firm with production function  $F$ . Hence, the firms' decisions in period  $t$  are determined by the solution to the problem

$$\text{maximize}_{K,L} F(A(t), K, L) - \sigma(t) \cdot K - \omega(t) \cdot L$$

where  $A(t)$ ,  $\sigma(t)$ , and  $\omega(t)$  are given and the price of output  $Y(t)$  in  $t$  is normalized to 1.

As  $F$  is concave, by the first order condition for maximization,  $\sigma(t) = F_K$  and  $\omega(t) = F_L$ . By Euler's theorem,  $Y(t) = \sigma(t) \cdot K(t) + \omega(t) \cdot L(t)$ , so profits are zero.

The Inada conditions (after Ken-Ichi Inada) state that

$$(a) \lim_{K \rightarrow 0} F_K = \infty \text{ and } \lim_{K \rightarrow \infty} F_K = 0$$

$$(b) \lim_{L \rightarrow 0} F_L = \infty \text{ and } \lim_{L \rightarrow \infty} F_L = 0.$$

Interpretation: the initial units of an input are highly productive but, for a sufficiently high amount of input, additional units are unproductive. Inada's conditions are helpful to ensure interior solutions ( $K > 0$  and  $L > 0$ ). They are not satisfied by production functions that are linear in  $K$  or  $L$  (like, for instance,  $F = A \cdot (K + L)$ ).

Fig. 1 next depicts what may be termed the economy's life cycle.

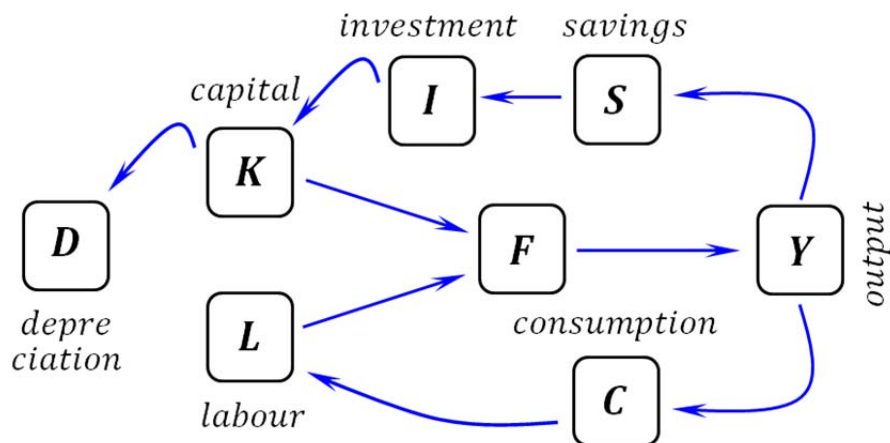


Fig. 1. Basic description of the economy represented by the model

The macroeconomic equilibrium condition is  $Y(t) = C(t) + I(t)$  or  $S(t) = I(t)$ , where  $S(t) = s \cdot Y(t)$ .

The capital stock accumulation condition is  $K(t + 1) = I(t) + (1 - \delta) \cdot K(t)$ .

To sum up, the model is given by the following equations (when the first four are inserted into the last one, the last equation summarizes the whole model).

- Aggregate production function  $Y(t) = F(A, K(t), L)$
- Uses of aggregate output  $Y(t) = C(t) + I(t)$
- Aggregate savings function  $S(t) = s \cdot Y(t)$ ,  $0 < s < 1$   
(and, therefore,  $C(t) = (1 - s) \cdot Y(t)$ )
- Macroeconomic equilibrium  $S(t) = I(t)$
- Capital stock accumulation  $\Delta K(t) = I(t) - \delta \cdot K(t) = s \cdot Y(t) - \delta \cdot K(t)$

Given population  $L$ , an initial capital stock  $K(1)$ , and technology represented by  $A$ , an equilibrium path for the economy is a sequence  $\{K(t), Y(t), C(t), \sigma(t), \omega(t)\}_{t \geq 1}$  such that:

- $K(t + 1) = s \cdot F(A, K(t), L) + (1 - \delta) \cdot K(t)$
- $Y(t) = F(A, K(t), L)$
- $C(t) = (1 - s) \cdot Y(t)$
- $\sigma(t) = F_K$
- $\omega(t) = F_L$ .

Knowing the sequence  $(K(1), K(2), K(3), \dots, K(t) \dots)$  of capital stocks, the rest of variables can be determined. Hence, solving the model is reduced to determining that sequence.

## 6.2. Analysis

It is more convenient to express the model in per capita terms. Define capital per capita (or capital-labour ratio) as  $k(t) = \frac{K(t)}{L}$  and output (or income) per capita as  $y(t) = \frac{Y(t)}{L}$ .

By constant returns,

$$y(t) = \frac{Y(t)}{L} = \frac{1}{L} F(A, K(t), L) = F\left(A, \frac{K(t)}{L}, \frac{L}{L}\right) = F(A, k(t), 1) = f(k(t))$$

This says that output per capita is a function of capital per capita. Therefore,

$$F = Y(t) = L \cdot y(t) = L \cdot f(k(t)) = L \cdot f\left(\frac{K(t)}{L}\right).$$

Given that  $\sigma(t) = F_K$ ,

$$\sigma(t) = F_K = \frac{\partial F}{\partial K(t)} = L \cdot f'\left(\frac{K(t)}{L}\right) \cdot \frac{1}{L} = f'\left(\frac{K(t)}{L}\right) = f'(k(t)).$$

Consequently, the marginal product of  $f$  coincides with the price of capital. And since  $F_K > 0$ , it follows that  $f' > 0$ : the output per capita function  $f$  is increasing.

By assumption,  $F_{KK} := \frac{\partial^2 F}{\partial K^2} < 0$ . In view of this, as  $F = L \cdot f\left(\frac{K(t)}{L}\right)$ ,

$$0 > F_{KK} := \frac{\partial^2 F}{\partial K^2} = \frac{\partial \left( f'\left(\frac{K(t)}{L}\right) \right)}{\partial K} = \frac{1}{L} \cdot f''\left(\frac{K(t)}{L}\right)$$

which implies  $f'' < 0$ . Accordingly,  $f$  is a concave (increasing) function.

By Euler's theorem,  $F = F_K \cdot K(t) + F_L \cdot L$ . By the profit maximizing assumption,  $F_K = \sigma(t)$  and  $F_L = \omega(t)$ . As just shown,  $\sigma(t) = f'(k(t))$ . Summarizing

$$F(A, K(t), L) = f'(k(t)) \cdot K(t) + \omega(t) \cdot L.$$

If both sides are divided by  $L$ ,

$$y(t) = f'(k(t)) \cdot k(t) + \omega(t).$$

Solving for  $\omega(t)$  yields

$$\omega(t) = f(k(t)) - k(t) \cdot f'(k(t))$$

The dynamics of capital accumulation is represented by the equation  $K(t+1) = s \cdot F(A, K(t), L) + (1 - \delta) \cdot K(t)$ . If both sides are divided by  $L$ ,

$$\boxed{k(t+1) = s \cdot f(k(t)) + (1 - \delta) \cdot k(t)} \tag{1}$$

which represents the dynamics of capital per capita accumulation. Equation (1) summarizes the model when there is neither population growth nor technological progress.

A steady-state equilibrium (without population growth nor technological progress) is an equilibrium path where, for some  $\bar{k}$  and all  $t$ ,  $k(t) = \bar{k}$  (capital per capita remains constant).

Being  $L$  constant, having  $k$  constant implies that  $K$  is constant (hence,  $Y$  and  $y$  are also constant).

Steady-state equilibria are obtained from (1) by setting  $k(t + 1) = k(t) = \bar{k}$ . Thus,  $\bar{k} \neq 0$  satisfies

$$\frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{s}$$

**Proposition 1.** Given the assumptions on  $F$ , there is a unique  $\bar{k} \neq 0$  such that  $\frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{s}$ , with output per capita  $\bar{y} = f(\bar{k})$  and consumption per capita  $\bar{c} = (1 - s) \cdot f(\bar{k})$ .

Fig. 2 represents (1) and the only steady-state equilibrium  $\bar{k} > 0$ .

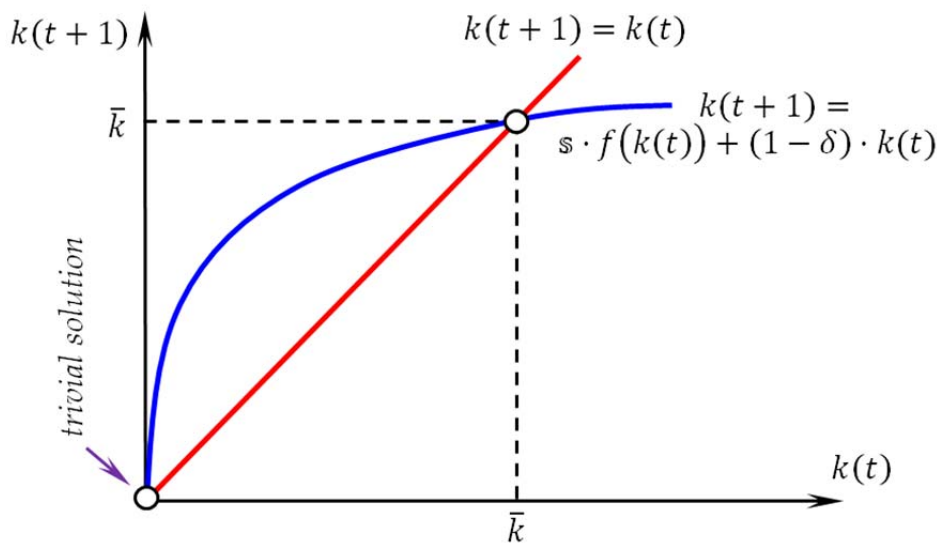


Fig. 2. Graphical representation of the model and its steady-state equilibria

The value  $\bar{k}$  synthesizes the solution of the model. On the one hand, as shown below, any other positive value for  $k$  is unstable. And, on the other, the rest of variables can be determined once  $\bar{k}$  is known. Specifically,  $\bar{y} = f(\bar{k})$ ,  $\bar{s} = s \cdot f(\bar{k})$ ,  $\bar{K} = \bar{k} \cdot L$ ,  $\bar{Y} = f(\bar{k}) \cdot L$ ,  $\bar{c} = \bar{y} - \bar{s} = f(\bar{k}) - s \cdot f(\bar{k})$ ,  $\bar{i} = \bar{d} = \delta \cdot \bar{k} \dots$

Fig. 3 next provides an alternative graphical representation of the model, based on displaying separately the (per capita) production function  $f$ , the savings function  $s \cdot f$ , and the depreciation function  $\delta \cdot k$ .

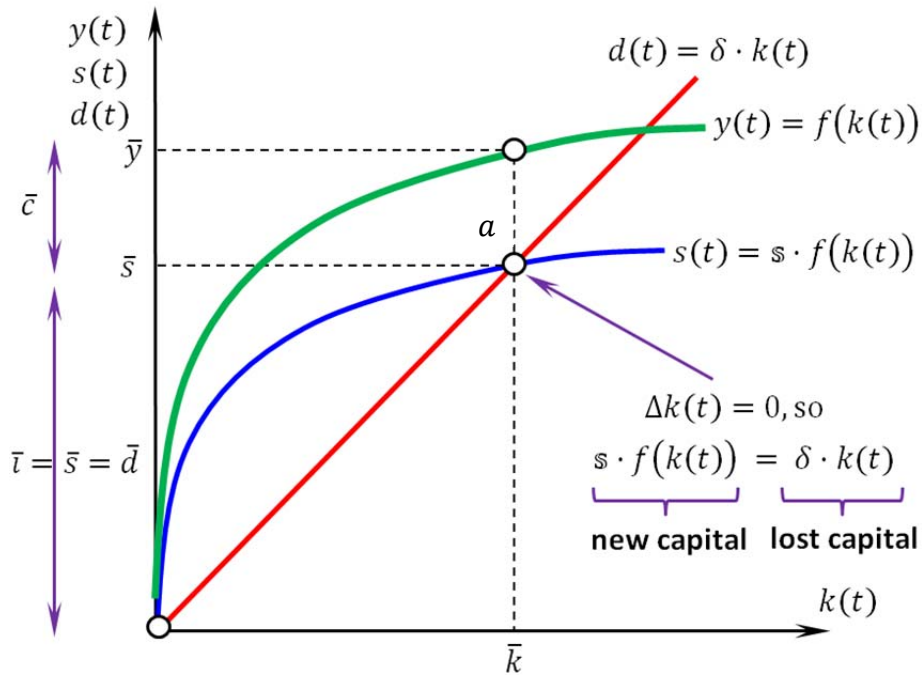


Fig. 3. Alternative graphical representation of the model and its steady-state equilibria

In a steady-state equilibrium, the curves representing the new capital ( $s \cdot f$ ) and the lost capital ( $\delta \cdot k$ ) intersect. This intersection is point  $a$  in Fig. 3.

Fig. 4 illustrates the possibility of non-existence and non-uniqueness of the non-trivial steady-state equilibrium. In case (i),  $f$  fails to be concave (marginal productivities need not be decreasing). Several interior steady states may exist. Case (ii) shows that the failure of the Inada conditions may lead to the non-existence of interior steady states.

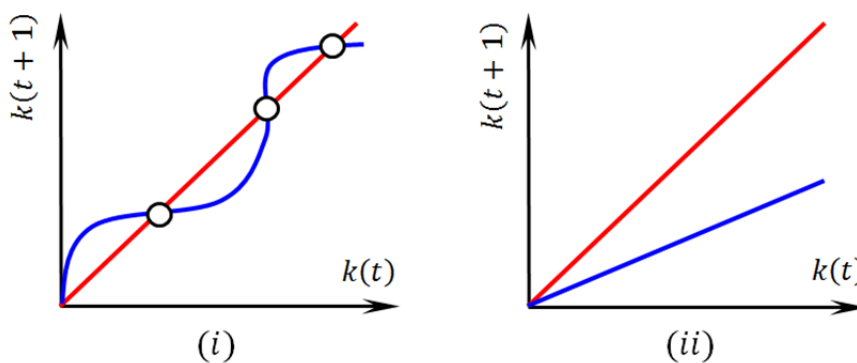


Fig. 4. Failure of uniqueness and existence

**Proposition 2.** Given the assumptions on  $F$ :

- (i) the steady-state equilibrium of  $k(t + 1) = s \cdot f(k(t)) + (1 - \delta) \cdot k(t)$ , which corresponds to  $k = \bar{k}$ , is globally asymptotically stable, which means that  $\{k(t)\}$  converges to  $\bar{k}$ ; and
- (ii) for all  $k(1) > 0$ ,  $\{k(t)\}$  converges monotonically to  $\bar{k}$ .

Hence, the dynamics of capital accumulation is similar to the dynamics in the OLG model.

Fig. 5 shows how capital per capita converges to  $\bar{k}$  when the initial value  $k(1)$  of capital per capita is smaller than  $\bar{k}$ . A similar reasoning applies when the initial value is larger than  $\bar{k}$ .

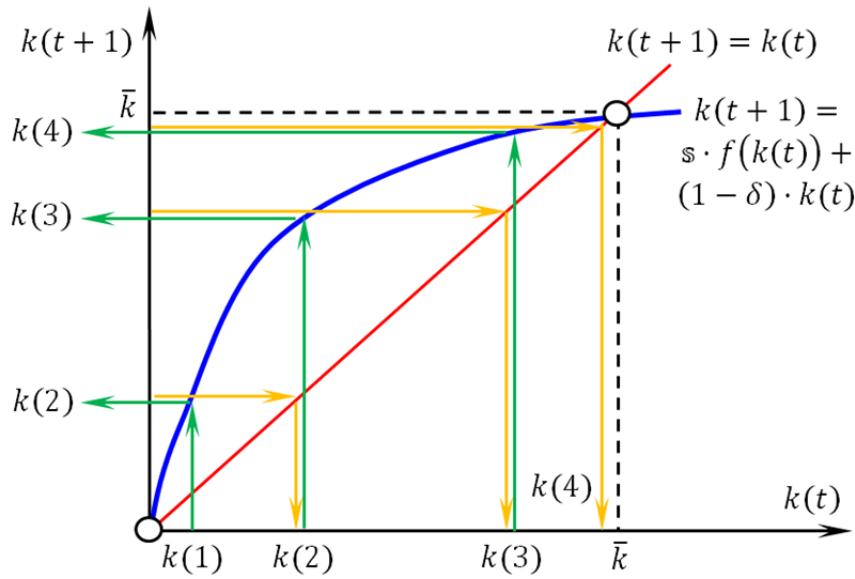


Fig. 5. Convergence to the interior steady-state equilibrium

It is worth noticing that, if  $k(1) < \bar{k}$ , then the sequence of wages  $\{\omega(t)\}_{t \geq 1}$  is increasing, whereas  $\{\sigma(t)\}_{t \geq 1}$  is a decreasing sequence ( $\uparrow k \Rightarrow \downarrow f' \Rightarrow \downarrow \sigma$ ). When  $k(1) < \bar{k}$ , convergence to  $\bar{k}$  means that capital accumulates ( $k$  increases). Since  $\uparrow k$  implies  $\uparrow f$  and  $\downarrow f'$ , it follows from  $\omega = f - k \cdot f'$  that  $\omega$  falls. Intuitively, more  $k$  makes labour more productive, so  $\omega$  rises.

If  $k(1) > \bar{k}$ , capital decumulates, which implies that  $\{\omega(t)\}_{t \geq 1}$  is decreasing and  $\{\sigma(t)\}_{t \geq 1}$  is increasing.

The following comparative statics analysis relies on the fact that  $\frac{f(k)}{k}$  is a decreasing function of  $k$ , since  $\frac{\partial \left( \frac{f(k)}{k} \right)}{\partial k} < 0$ .

- Effect of a change in the depreciation rate:  $\uparrow \delta \Rightarrow \downarrow \bar{k}$ .

$$\text{Given } \frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{s}, \uparrow \delta \Rightarrow \uparrow \frac{\delta}{s} \Rightarrow \uparrow \frac{f(\bar{k})}{\bar{k}} \Rightarrow \downarrow \bar{k}.$$

- Effect of a change in the savings rate:  $\uparrow s \Rightarrow \uparrow \bar{k}$ .

$$\text{Given } \frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{s}, \uparrow s \Rightarrow \downarrow \frac{\delta}{s} \Rightarrow \downarrow \frac{f(\bar{k})}{\bar{k}} \Rightarrow \uparrow \bar{k}.$$

- Effect of a technological improvement: a shift upwards of  $f$  to  $g$  yields  $\uparrow \bar{k}$ .

$$\text{Given } \frac{g(\bar{k}_g)}{\bar{k}_g} = \frac{\delta}{s} = \frac{f(\bar{k}_f)}{\bar{k}_f}, \frac{g(\bar{k}_f)}{\bar{k}_f} > \frac{\delta}{s} \text{ follows from } g(\bar{k}_f) > f(\bar{k}_f),$$

$$\text{so } \uparrow \bar{k} \text{ is needed to get } \downarrow \frac{g(\bar{k}_f)}{\bar{k}_f}. \text{ Therefore, } \bar{k}_g > \bar{k}_f.$$

Letting  $\Delta k(t) := k(t + 1) - k(t)$ , equation (1) can be equivalently expressed as

$$\Delta k(t) = s \cdot f(k(t)) - \delta \cdot k(t).$$

Then the growth rate  $g_{k(t)}$  of  $k(t)$  is

$$g_{k(t)} = \frac{\Delta k(t)}{k(t)} = s \cdot \frac{f(k(t))}{k(t)} - \delta = s \cdot \frac{y(t)}{k(t)} - \delta.$$

Therefore,  $g_{k(t)} > 0$  ( $k$  accumulates) if and only if  $s \cdot \frac{f(k)}{k} > \delta$ . With  $k(1) < \bar{k}$ , convergence to  $\bar{k}$  implies that  $g_{k(t)}$  falls and goes to 0. Fig. illustrates this.

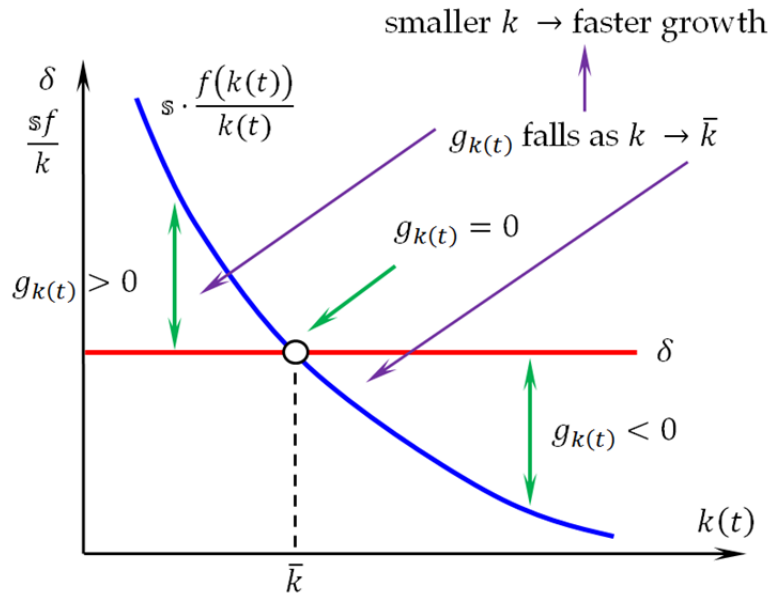


Fig. 6. Alternative illustration of the convergence to the interior steady-state equilibrium

### 6.3. The golden rule of capital accumulation

Steady-state per capita consumption  $\bar{c}$ , defined as a function of the savings rate  $s$ , is

$$\bar{c}(s) := (1 - s) \cdot f(\bar{k}(s)) = f(\bar{k}(s)) - s \cdot f(\bar{k}(s)) = f(\bar{k}(s)) - \delta \cdot \bar{k}(s).$$

The golden rule savings rate  $s^*$  maximizes steady-state per capita consumption  $\bar{c}$ .

$$0 = \frac{d\bar{c}(s)}{ds} = f'(\bar{k}(s)) \cdot \frac{d\bar{k}(s)}{ds} - \delta \cdot \frac{d\bar{k}(s)}{ds} = \frac{d\bar{k}(s)}{ds} \cdot (f'(\bar{k}(s)) - \delta).$$

Therefore,  $s^*$  satisfies  $f'(\bar{k}(s^*)) = \delta$ . The value  $\bar{k}(s^*)$  can be computed from  $f' = \delta$ . Given  $\bar{k}(s^*)$  and the steady-state condition  $s^* \cdot f(\bar{k}(s^*)) = \delta \cdot \bar{k}(s^*)$ , then  $s^*$  satisfies

$$s^* = \frac{\bar{k} \cdot f'(\bar{k})}{f(\bar{k})},$$

which is the share of  $k$  in  $y$  (in the steady-state equilibrium). Fig. 7 illustrates this.



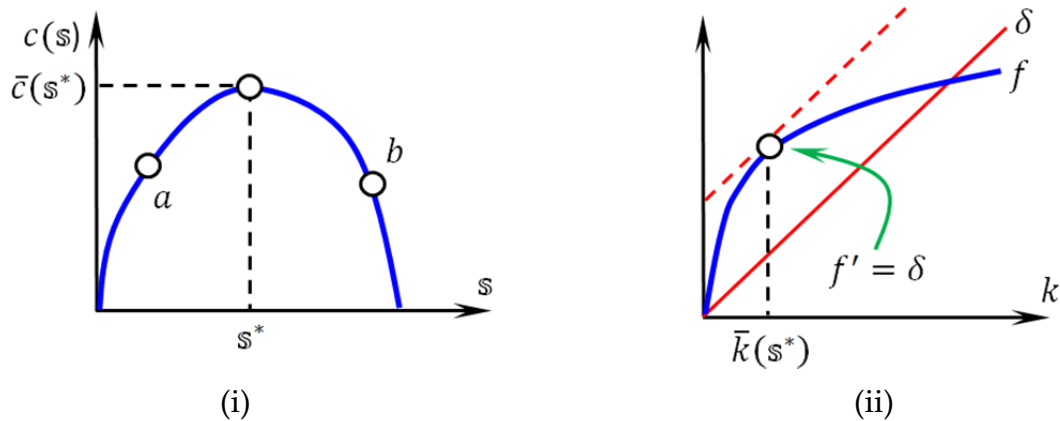


Fig. 7. The golden rule solution

The golden rule can be interpreted as deriving from an equilibrium in returns. Consider capital  $K$  and labour  $L$  as two assets produced in the economy.

The rate of return of capital can be defined as  $f' - \delta$ : the productivity of capital minus the depreciation of capital.

The rate of return of labour can be taken to be zero: population does not grow.

If both returns are to be the same, then  $f' - \delta = 0$ ; that is,  $f' = \delta$ .

An economy is dynamically inefficient if per capita consumption can be increased by reducing the capital stock (someone is better off, and no one worse off, with less capital).

**Proposition 3.** *If  $s > s^*$ , then (at the corresponding steady state) the economy is dynamically inefficient.*

Capital overaccumulation (point  $b$  in Fig. 7(i)) generates dynamic inefficiency ( $a$  does not). If  $s_b$  is reduced to  $s^*$ , per capita consumption is higher during the transition to the new steady state (immediate effect) and at the new steady state (permanent effect).

#### 6.4. Dynamics of capital accumulation with population growth

Let  $L(t + 1) = (1 + n) \cdot L(t)$ . The condition for capital stock accumulation is, as before,

$$K(t + 1) = s \cdot Y(t) + (1 - \delta) \cdot K(t).$$

Dividing both sides by  $L(t + 1)$ ,

$$k(t + 1) = s \cdot \frac{Y(t)}{L(t + 1)} + (1 - \delta) \cdot \frac{K(t)}{L(t + 1)}.$$

That is,

$$k(t + 1) = \frac{s}{1 + n} \cdot y(t) + \frac{1 - \delta}{1 + n} \cdot k(t)$$

or

$$\Delta k(t) = \frac{s}{1+n} \cdot f(k(t)) - \frac{\delta+n}{1+n} \cdot k(t).$$

In a steady state,  $\Delta k(t) = 0$ . Therefore, capital per capita  $\bar{k}$  in a steady state satisfies

$$\frac{f(\bar{k})}{\bar{k}} = \frac{\delta+n}{s}.$$

Despite the presence of the denominator  $1+n$ ,  $\bar{k}$  can be calculated by equating  $s \cdot y(t)$  with  $(\delta+n) \cdot k(t)$ ; see Fig. 8. The growth rate  $g_{k(t)}$  of  $k(t)$  is now; see Fig. 9.

$$g_{k(t)} = \frac{s}{1+n} \cdot \frac{f(k(t))}{k(t)} - \frac{\delta+n}{1+n}.$$

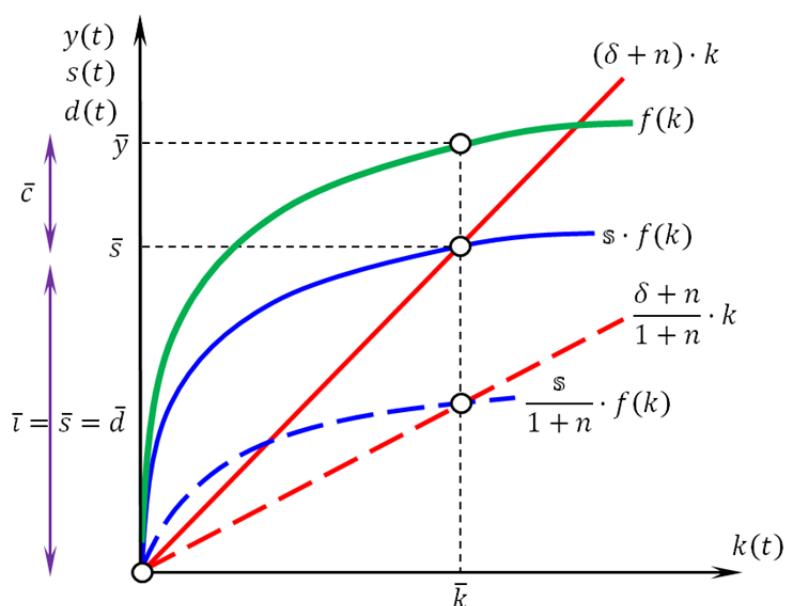


Fig. 8. Steady-state equilibrium with population growth

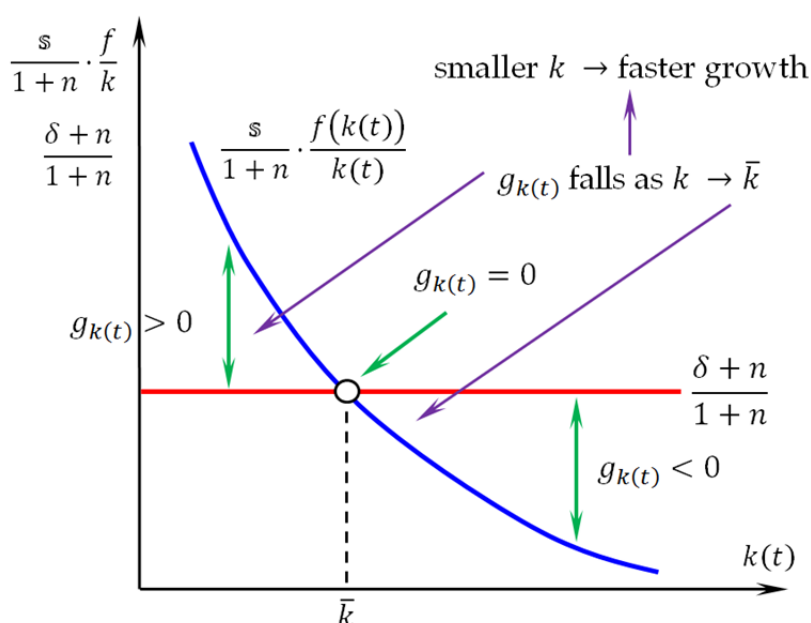


Fig. 9. Growth rates with population growth

In a steady state, no matter whether  $n = 0$  or  $n > 0$ , per capita variables remain constant:  $y, k, c \dots$

But, when  $n = 0$ , aggregate variables ( $Y, K, C \dots$ ) also remain constant, because  $L$  is constant:  $k$  and  $L$  constant imply  $K$  constant,  $y$  and  $L$  constant imply  $Y$  constant, etc.

When  $n > 0$ ,  $k = K/L$  constant and  $L$  growing at rate  $n$  imply  $K$  growing at rate  $n$ . By constant returns, if  $K$  and  $L$  grow at rate  $n$ , then  $Y$  also grows at rate  $n$ .

To determine the golden rule solution with population growth, in a steady state, consumption per capita is

$$\bar{c}(s) := (1 - s) \cdot f(\bar{k}(s)) = f(\bar{k}(s)) - s \cdot f(\bar{k}(s)) = f(\bar{k}(s)) - (\delta + n) \cdot \bar{k}(s).$$

The golden rule savings rate  $s^*$  maximizes steady-state per capita consumption  $\bar{c}$ .

$$0 = \frac{d\bar{c}(s)}{ds} = f'(\bar{k}(s)) \cdot \frac{d\bar{k}(s)}{ds} - (\delta + n) \cdot \frac{d\bar{k}(s)}{ds} = \frac{d\bar{k}(s)}{ds} (f'(\bar{k}(s)) - (\delta + n))$$

Therefore,  $s^*$  satisfies

$$f'(\bar{k}(s^*)) = \delta + n.$$

$\bar{k}(s^*)$  can be computed from  $f' = \delta + n$ . Using  $\bar{k}(s^*)$  and the steady-state condition

$$s^* \cdot f(\bar{k}(s^*)) = (\delta + n) \cdot \bar{k}(s^*)$$

it follows that  $s^*$  satisfies

$$s^* = \frac{\bar{k} \cdot f'(\bar{k})}{f(\bar{k})}.$$

The situation is analogous to the one arising when  $n = 0$ ; see Fig. 10.

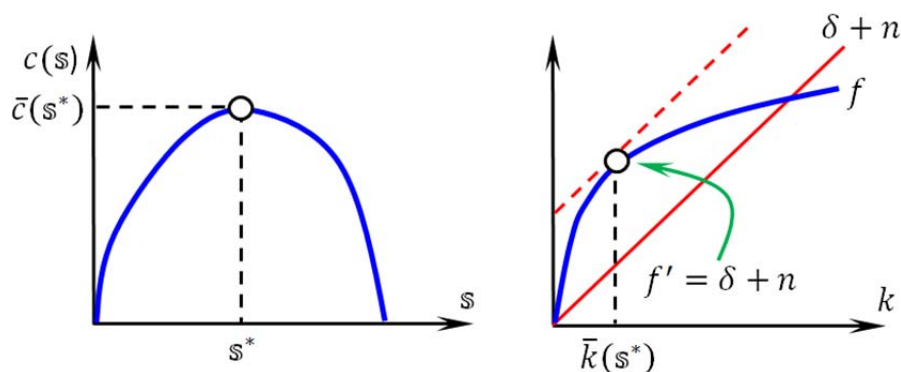


Fig. 10. The golden rule solution with population growth

## 6.5. Dynamics of capital accumulation with population growth and technological progress

For a production function of the sort  $F(A, K, L)$ , there are three basic types of technological progress.

- Hicks-neutral technological progress means that  $F(A, K, L)$  can be expressed, for some  $G$ , as  $A \cdot G(K, L)$ . A technological change just relabels isoquants (no change in shape).
- Solow-neutral (or capital-augmenting) technological progress means that  $F(A, K, L)$  can be written as  $G(A \cdot K, L)$ , for some  $G$ . Higher  $A$  is equivalent to having more  $K$ .
- Harrod-neutral (or labor-augmenting) technological progress makes  $F(A, K, L)$  equal to  $G(K, A \cdot L)$ , for some  $G$  (higher  $A$  is like more  $L$ )

Suppose population grows at rate  $n$  and technology accumulates at a constant rate  $a$ :  $L(t + 1) = (1 + n) \cdot L(t)$  and  $A(t + 1) = (1 + a) \cdot A(t)$ .

Technological progress is assumed to be Harrod-neutral. Define per capita variables in terms of efficiency units of labour  $A \cdot L$  (also called "effective labour units"):

$$k(t) = \frac{K(t)}{A(t) \cdot L(t)} \quad y(t) = \frac{Y(t)}{A(t) \cdot L(t)}.$$

The capital stock accumulates following the usual rule

$$K(t + 1) = s \cdot Y(t) + (1 - \delta) \cdot K(t).$$

After dividing both sides by  $A(t + 1) \cdot L(t + 1)$ ,

$$\begin{aligned} k(t + 1) &= \frac{s \cdot Y(t)}{A(t + 1) \cdot L(t + 1)} + \frac{(1 - \delta) \cdot K(t)}{A(t + 1) \cdot L(t + 1)} = \\ &= \frac{s}{(1 + a) \cdot (1 + n)} \cdot \frac{Y(t)}{A(t) \cdot L(t)} + \frac{1 - \delta}{(1 + a) \cdot (1 + n)} \cdot \frac{K(t)}{A(t) \cdot L(t)} = \\ &= \frac{s}{(1 + a) \cdot (1 + n)} \cdot y(t) + \frac{1 - \delta}{(1 + a) \cdot (1 + n)} \cdot k(t) \end{aligned}$$

In sum,

$$k(t + 1) = \frac{s}{(1 + a) \cdot (1 + n)} \cdot y(t) + \frac{1 - \delta}{(1 + a) \cdot (1 + n)} \cdot k(t)$$

or, equivalently,

$$\Delta k(t) = \frac{s}{(1 + a) \cdot (1 + n)} \cdot f(k(t)) - \frac{\delta + n + a(1 + n)}{(1 + a) \cdot (1 + n)} \cdot k(t).$$

In a steady state,  $\Delta k(t) = 0$ . Accordingly, capital per capita  $\bar{k}$  in a steady state satisfies

$$\boxed{\frac{f(\bar{k})}{\bar{k}} = \frac{\delta + n + a \cdot (1 + n)}{s}}.$$

Despite the presence of  $(1+a) \cdot (1+n)$  in the denominator,  $\bar{k}$  can be obtained by equating  $s \cdot f(k(t))$  with  $(\delta + n + a(1+n)) \cdot k(t)$ ; see Fig. 11. The growth rate  $g_{k(t)}$  of  $k(t)$  is now

$$g_{k(t)} = \frac{s}{(1+a) \cdot (1+n)} \cdot \frac{f(k(t))}{k(t)} - \frac{\delta + n + a \cdot (1+n)}{(1+a) \cdot (1+n)}. \quad (2)$$

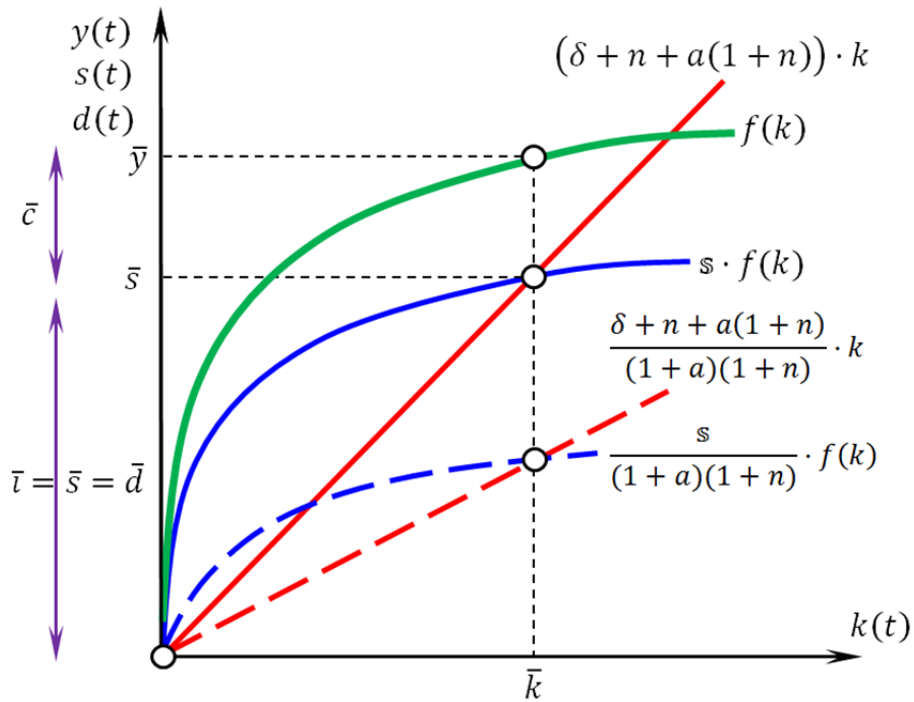


Fig. 11. Steady-state equilibrium with population growth and technological progress

Technological progress produces the following effect: as  $k$  converges to  $\bar{k}$ , output per efficiency unit of labour  $y$  also converges (to some  $\bar{y}$ ), so output per efficiency unit does not grow in the long run. Formally,  $\frac{K}{A \cdot L}$  is eventually constant ( $\bar{k}$ ). Denoting growth rates by  $g$ , that  $\frac{K}{A \cdot L}$  is constant implies that  $g_K - (g_A + g_L) \approx 0$ . That is,  $g_K - g_L \approx g_A$ . Summing up,

$$g_{K/L} \approx g_A.$$

Output per person grows at the same rate as technology: technological progress offsets the diminishing returns to capital.

Summarizing, the Solow-Swan model exhibits the following characteristics.

- There is a unique (non-trivial) steady state.
- The steady state is asymptotically stable and comparative statics is simple.
- There is no sustained growth: growth occurs only when the economy shifts from one steady state to another steady state with a higher  $\bar{k}$ .
- Sustained growth requires sustained technological change.
- The rate of growth of output (long term growth) is not affected by  $s$ : it only affects the levels of  $Y$ ,  $K$ , and  $C$  (more savings only depress the marginal productivity of capital).

## 6.6. Convergence

Convergence (or catch-up effect) is the hypothesis according to which there is a tendency for output per capita in poorer economies to grow faster than output per capita in richer economies.

Absolute (unconditional) convergence holds that the catch up occurs regardless of the characteristics of the economies. There are two basic reasons to justify the existence of absolute convergence.

- (i) Technological transfer. The “advantage of backwardness”: poorer economy can replicate technology, production methods, and institutions developed in their first place by richer economies without having to bear the cost of creating or developing them.
- (ii) Capital accumulation. Increasing accumulation from a low level has a bigger impact because diminishing returns in poorer economies are not as strong as in richer economies.

The empirical evidence does not yet offer evidence for absolute convergence (at least Africa is not catching up with the rest of the world); see Fig. 12(a) and 12(b). For convergence to occur in a given time interval, it is to be expected a negative relationship between output per capita at the start of the interval and the average growth rate of output per capita in that interval.

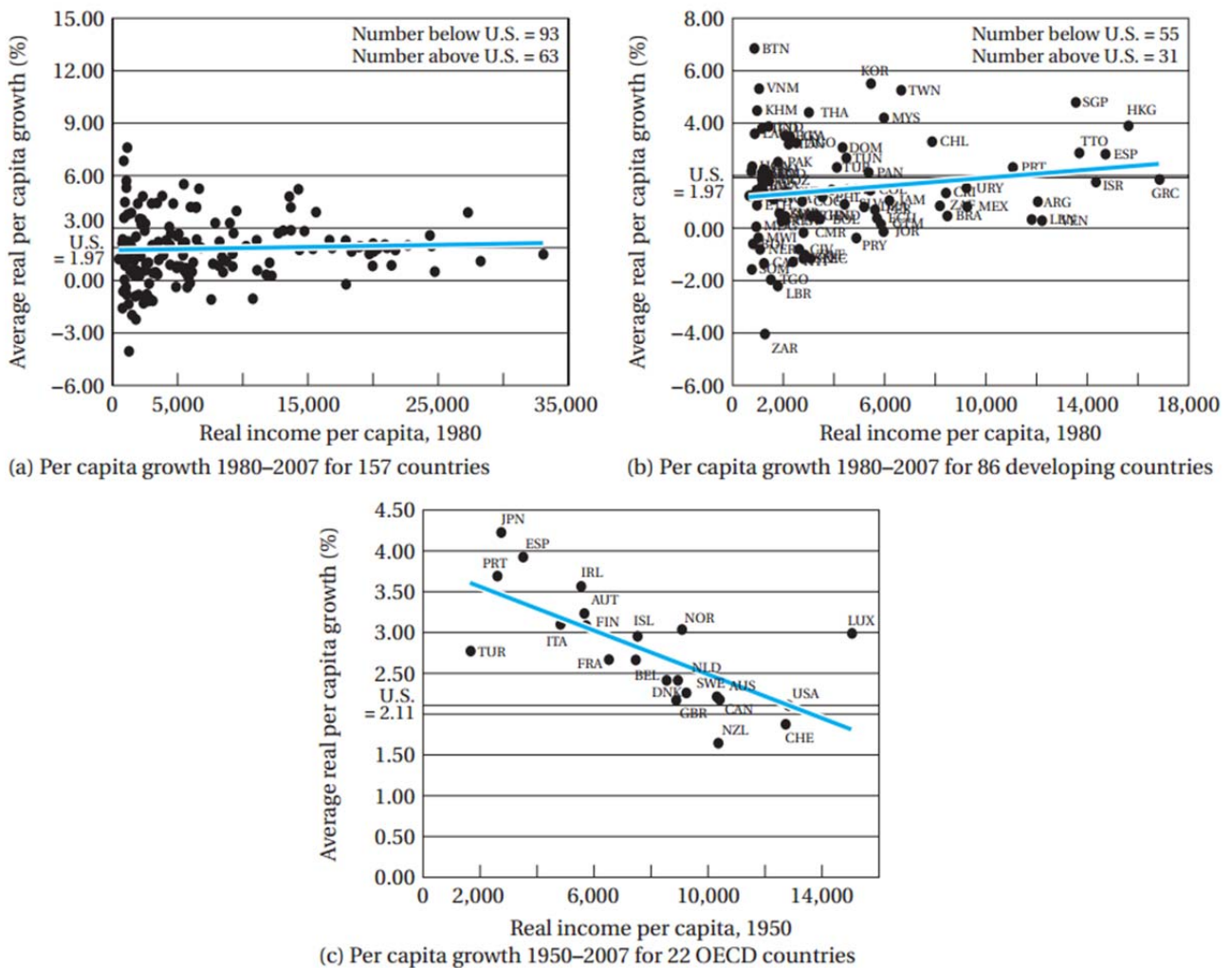


Fig. 12. Conditional convergence (MP Todaro, SC Smith (2011): *Economic development*, p. 80)

Problem of selection bias: convergence is tested for rich countries so, for those who were initially relatively poor, it is plain that they had to grow faster than the rich countries initially rich.



The Lucas paradox (Robert E. Lucas, Jr, 1990) refers to the claim that capital is apparently not flowing from richer to poorer despite the fact that poorer economies have lower levels of capital per capita, which should correspond to higher returns on capital and, therefore, make investment more attractive in poorer countries. Recent research suggests that, for the period 1970-2000, the leading explanation for the lack of capital flows to poorer countries is low institutional quality.

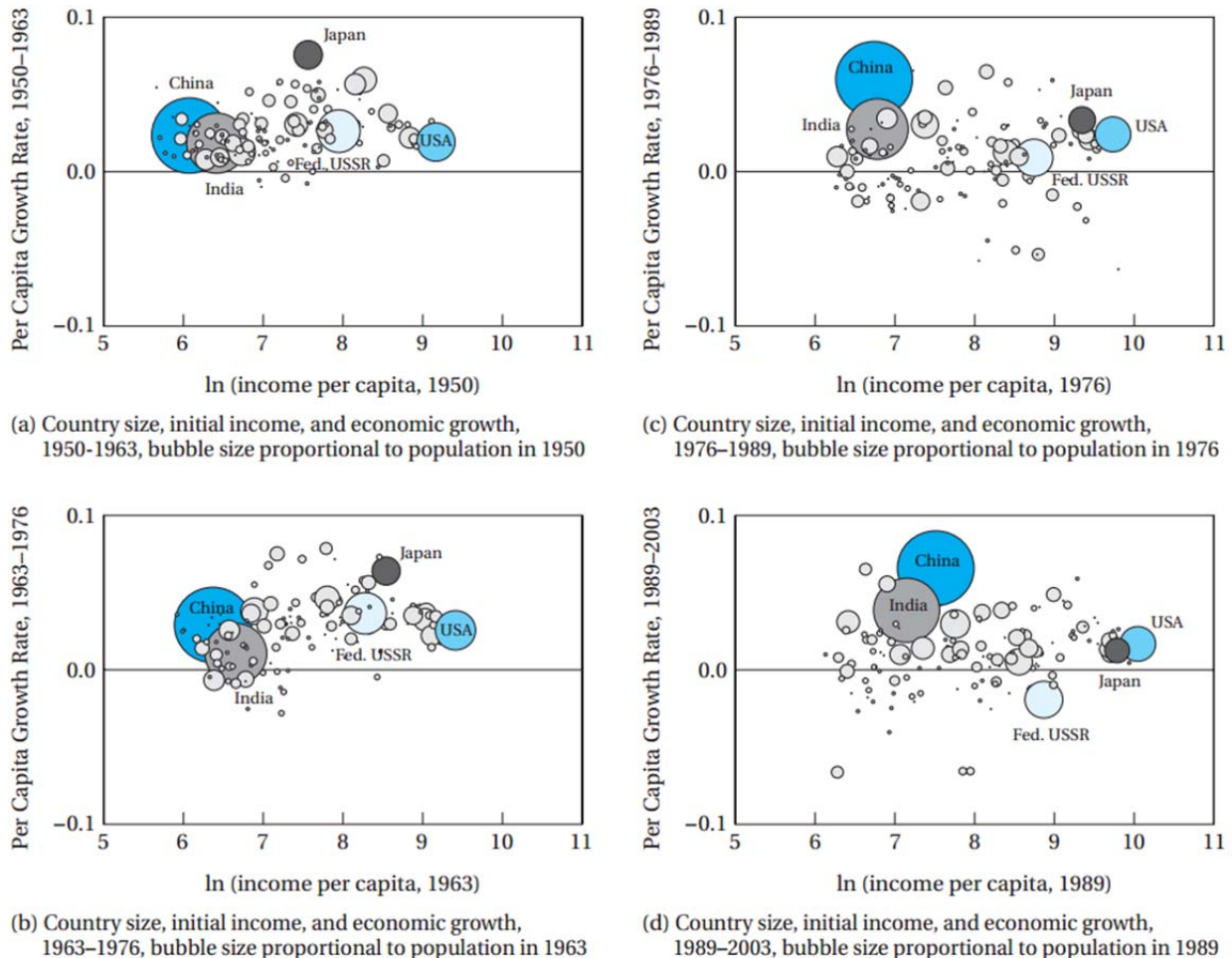


Fig. 13. Convergence as a function of country size  
(MP Todaro, SC Smith (2011): *Economic development*, p. 82)

The concept of conditional convergence means that poorer economies grow faster if they have the same technology and fundamental variables for capital accumulation ( $s, \delta, n$ ) as richer economies.

Conditional convergence occurs in the SS model: economies with the same technology and fundamental variables converge to the same steady state (similar countries converge). In fact, by (2), the SS model predicts that poorer countries would grow faster than richer economies. In a poor economy,  $k$  is comparatively small. Therefore,  $f(k)/k$  is high. Given the rest of parameters ( $s, a, n$ , and  $\delta$ ), high  $f(k)/k$  implies, by (2), high  $g_k$ . This, in turn, implies high  $g_y$ .

But since marginal productivities are decreasing, growth cannot last. In the SS model, (per capita) growth occurs only while the economy converges to the steady state. Once there, per capita variables remain stagnant. The empirically observed convergence has been lower than predicted by the SS model. Convergence: OECD, Asian tigers. Divergence: Africa.

According to the hypothesis of conditional convergence, output per capita in an economy converges to a certain level dictated by the structural characteristics of the economy. Given that what guides convergence is not the initial value of output per capita, there is no need to transfer income from richer to poorer nations: foreign aid should concentrate on establishing the necessary structural characteristics (related to, for instance, the ability to absorb new technology, attract capital and participate in global markets).

The concept of strong conditional convergence expresses convergence conditional on having the same determinants of capital accumulation:  $s$ ,  $\delta$ , and  $n$ . It relies on the presumption that any technology is eventually available to all economies. What separates economies is capital accumulation not technology.

The term sigma-convergence refers to a reduction in the dispersion of output per capita levels across economies. Beta-convergence is said to occur when poorer economies grow faster than richer ones. Conditional beta-convergence means beta-convergence conditional on certain variables held constant. Inconditional (or absolute) beta-convergence" takes place when the growth rate of an economy declines as it approaches its steady state; see [http://en.wikipedia.org/wiki/Convergence\\_%28economics%29](http://en.wikipedia.org/wiki/Convergence_%28economics%29).

Convergence in terms of rates contends that differences in growth rates (mainly,  $y$ ) tend to vanish. It does not imply convergence in the level of output per capita. There is convergence in rates in the SS model between economies with the same technology: the same rate of technological change yields the same growth rate of output per capita (convergence club = those who innovate).

The expression club convergence captures the observation that one can identify groups of economies with similar growth trajectories. In particular, there are poor (low income) economies that simultaneously experience low growth rates. Certain structural barriers (limited access to education, lack of resources, poor infrastructure) may prevent a poor economy from migrating to a richer convergence club. Consequently, it may prove extremely difficult to move from one to another convergence club.

The SS model displays a negative correlation between  $\alpha$  and the speed of convergence.

The lower  $\alpha$ , the more decreasing returns to capital are. This means that, to converge, capital should accumulate faster.

With more capital, rates of returns are lower. Capital migrates to poorer economies, where rates of return are higher. Through diffusion of knowledge, poorer economies can adopt technological advances developed in richer economies.



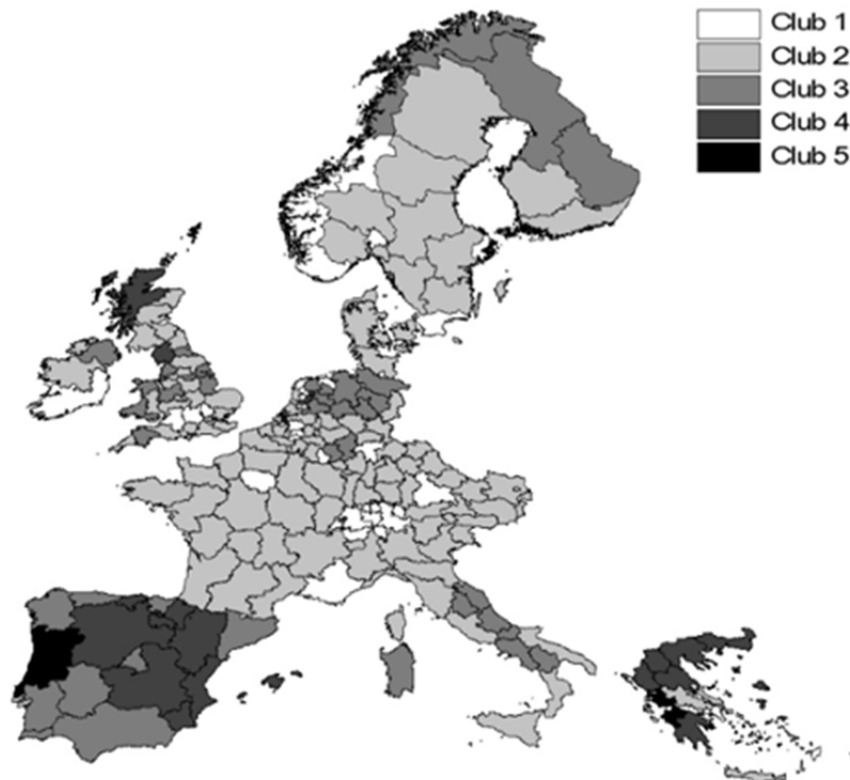


Fig. 14. Convergence club, Europe, 1990-2005 · <http://www.ub.edu/sea2009.com/Papers/129.pdf>

## 6.7. Stylized facts of growth

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The following are “Remarkable historical constancies revealed by recent empirical investigation” according to Nicholas Kaldor (1957, 1961).

1.  $Y/L$  grows (at a roughly constant rate)
2.  $K/L$  grows continuously (follows from 1 and 4)
3. Rate of return  $\sigma$  on capital stable over the long run
4.  $K/Y$  has been roughly stable for long periods
5. The shares of  $K$  and  $L$  in output have remained approximately constant
6. Real wage grows over time (follows from 2, 4, and 5)
7. Output and productivity growth differ substantially across countries

Alternatively:

- “Labor productivity has grown at a sustained rate.
- Capital per worker has also grown at a sustained rate.
- The real interest rate or return on capital has been stable.
- The ratio of capital to output has also been stable.
- Capital and labor have captured stable shares of national income.
- Among the fast growing countries of the world, there is an appreciable variation in the rate of growth “of the order of 2–5 percent.”

Paul Romer (1989) suggested more facts.

8. In cross-section, the mean growth rate shows no variation with income per capita levels.
9. The rate of growth of inputs is insufficient to explain the growth of output.
10. Growth in the volume of trade is positively correlated with growth in output.
11. Population growth rates are negatively correlated with the level of income.
12. Skilled and unskilled workers tend to migrate towards high-income countries

New facts proposed by Charles I. Jones and Paul Romer (2009); see <http://www.stanford.edu/~chadj/Kaldor200.pdf>.

1. **Increases in the extent of the market.** Increased flows of goods, ideas, finance, and people — via globalization as well as urbanization — have increased the extent of the market for all workers and consumers.
2. **Accelerating growth.** For thousands of years, growth in both population and per capita GDP has accelerated, rising from virtually zero to the relatively rapid rates observed in the last century.
3. **Variation in modern growth rates.** The variation in the rate of growth of per capita GDP increases with the distance from the technology frontier.
4. **Large income and TFP differences.** Differences in measured inputs explain less than half of the enormous cross country differences in per capita GDP.
5. **Increases in human capital per worker.** Human capital per worker is rising dramatically throughout the world.
6. **Long-run stability of relative wages.** The rising quantity of human capital relative to unskilled labor has not been matched by a sustained decline in its relative price.

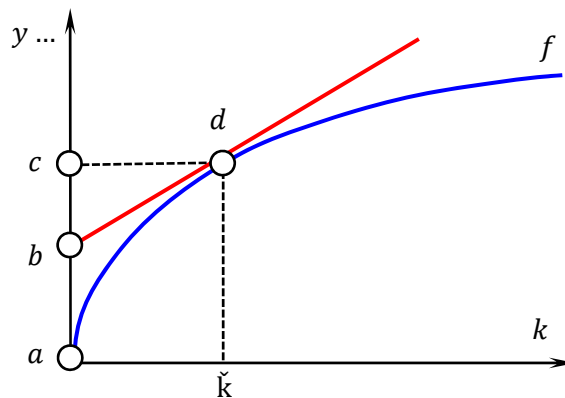
## Exercicis

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**Exercici 1. Teorema d'Euler.** Demuestra el Teorema d'Euler. [Suggeriment: partint de la definició d'homogeneïtat de grau  $h$ , deriva tots dos costats de l'equació respecte del paràmetre  $\lambda$  i considera el valor  $\lambda = 1$ ; per a la segona part del teorema, deriva ara respecte de  $K$  i divideix per  $\lambda$ .]

**Exercici 2. Cobb-Douglas.** Considera la funció Cobb-Douglas  $Y = F(A, K, L) = A \cdot K^\alpha \cdot L^{1-\alpha}$ , amb  $A > 0$  i  $0 < \alpha < 1$ . (i) Comprova si se satisfan totes les condicions imposades sobre la funció de producció en el model (productivitats dels factors, rendiments d'escala, condicions d'Inada). (ii) Definint  $y = \frac{Y}{L}$  i  $k = \frac{K}{L}$ , obté l'expressió corresponent  $y = f(k)$  (això és, determina  $f$ ). (iii) Assumint mercats competitius de  $K$  i  $L$ , indica la fórmula que expressa  $\sigma$  en termes de  $k$  i la que expressa  $\omega$  en termes de  $k$ .

**Exercici 3. Pagaments als factors.** Demuestra que, a la següent gràfica, quan l'estoc de capital per càpita és  $\bar{k}$  (la distància  $cd$ ), aleshores la distància  $ab$  representa el salari i la distància  $bc$  representa el benefici per treballador  $\sigma \cdot \bar{k}$ .



**Exercici 4. Funció  $f(k)/k$ .** En el cas sense creixement de la població ni progrés tecnològic, demostra que  $\frac{f(k)}{k}$  es una funció decreixent de  $k$ .

**Exercici 5. Preus dels factors.** (i) Suposant que no hi ha creixement de la població ni progrés tecnològic, explica perquè, si  $k(1) < \bar{k}$ , la seqüència de salaris  $\{\omega(t)\}$  és una seqüència creixent i la seqüència  $\{\sigma(t)\}$  de preus del factor capital és decreixent ( $\bar{k} \neq 0$  és el valor de  $k$  en l'estat estacionari). (ii) Mostra que  $\{\omega(t)\}$  és decreixent i  $\{\sigma(t)\}$  creixent si  $k(1) > \bar{k}$ .

**Exercici 6. Model de Solow i Swan estàndard.** Considera el model de Solow i Swan sense creixement de la població ni progrés tecnològic amb  $y = k^{1/3}$ , taxa de depreciació  $1/10$  i tasa d'estalvi  $2/5$ . (i) Calcula els valor de la producció per càpita, el consum per càpita, la inversió per càpita i la depreciació per càpita en l'estat estacionari. Assenyala aquests valors en una representació gràfica del model. (ii) Indica en la representació gràfica com variarien els valors calculats a (i) si: (a) la taxa de depreciació augmentés; (b) la taxa de d'estalvi s'incrementés; (c) es produïssin simultàniament (a) i (b). (iii) Obté la taxa d'estalvi i el consum per càpita corresponents a la regla d'or. És l'economia dinàmicament ineficient si inicialment  $k$  és 20? (iv) Representa gràficament la taxa de creixement de  $k$  en funció de  $k$ . (v) Respon a les preguntes anteriors si la taxa de creixement de la població (en tant per u) és  $1/10$ .

**Exercici 7. Anàlisi gràfica d'estàtica comparativa.** Mostra gràficament l'efecte sobre el capital per càpita  $\bar{k}$  d'estat estacionari (així com de la producció per càpita i el consum per càpita d'estat estacionari) si: (i) augmenta  $\delta$ ; (ii) augmenta  $s$ ; (iii) progrés tecnològic; (iv) tenen lloc simultàniament (ii) i (iii); (v) tenen lloc simultàniament (i) i (ii); (vi) disminueix  $\delta$  i augmenta  $s$ ; i (vii) es produeix un progrés tecnològic al mateix temps que disminueix  $s$ .

**Exercici 8. Progrés tecnològic.** Verifica que els tres tipus de progrés tecnològic neutral (en el sentit de Harrod, de Hicks i de Solow) són equivalents en la funció de producció Cobb-Douglas  $F(A, K, L) = A \cdot K^\alpha \cdot L^{1-\alpha}$ , amb  $A > 0$  i  $0 < \alpha < 1$ .

**Exercici 9. Model de Solow i Swan estàndard.** Considera el model de Solow i Swan sense progrés tecnològic on la població creix a la taxa (neta)  $n = \frac{1}{16}$  però amb la següent variant: la taxa d'estalvi és  $\frac{1}{4}$  si l'estoc de capital per càpita és superior a 3 i és  $\frac{1}{2}$  si l'estoc de capital per càpita és igual o inferior a 3. La funció de producció per càpita és  $f(k_t) = k_t^{1/2}$ . La taxa de depreciació és  $\frac{3}{16}$ . (i) Calcula el valors de totes les variables endògenes a tots els estats estacionaris on el capital per càpita és positiu. (ii) Assenayala els valors obtinguts en l'apartat (i) a una representació gràfica del model i explica si els estats estacionaris són estables.

**Exercici 10. Progrés tecnològic i creixement de la població.** Considera el model de Solow amb progrés tecnològic i creixement de la població amb funció de producció  $Y(t) = A(t) \cdot K(t)^{1/2} \cdot L(t)^{1/2}$ . La població creix a la taxa constant  $n > 0$  i la tecnologia s'acumula a la taxa constant  $a > 0$ , de manera que  $L(t + 1) = (1 + n) \cdot L(t)$  i  $A(t + 1) = (1 + a) \cdot A(t)$ . Defineix el capital per càpita com  $k(t) = \frac{K(t)}{A(t) \cdot L(t)}$  i la producció per càpita com  $y(t) = \frac{Y(t)}{A(t) \cdot L(t)}$ . (i) Determina l'equació en diferències que estableix la dinàmica del capital per càpita, la fórmula del capital per càpita en l'estat estacionari i la fórmula de la taxa d'estalvi que satisfà la regla d'or. (ii) Respon a les mateixes preguntes si  $a = n$ .

**Exercici 11. Funció de producció.** Verifica que, per a una funció de producció Cobb-Douglas  $Y = F(K, L)$ ,  $F_{LL} \cdot L + F_{LK} \cdot K = 0$ . Demuestra que, per a l'expressió per càpita  $f$  d' $F$ ,  $\sigma = f'$  i  $\omega = f - k \cdot f'$ .

**Exercici 12. Estabilitat.** Considera el model de Solow i Swan sense creixement de la població i sense progrés tecnològic on la depreciació  $d$  del capital per càpita, en comptes de ser la funció  $d = \delta \cdot k$ , és la funció  $d = 3/4$  si  $k < 1$  i  $d = 2/3 + k/12$  si  $k \geq 1$ . La taxa d'estalvi és  $1/2$  i la funció de producció agregada és  $Y = K^{1/2} \cdot L^{1/2}$ . (i) Representa gràficament les funcions per càpita. (ii) Calcula tots els estats estacionaris. (iii) Determina quins dels estats estacionaris són estables.

**Exercici 13. Estabilitat.** En el model de Solow i Swan sense creixement de la població i sense progrés tecnològic la taxa depreciació  $\delta$  del capital per càpita, en comptes de ser una constant, és la funció decreixent  $\delta = 1/k^2$ . Representa gràficament el model, identifica a la gràfica els estats estacionaris i explica quins dels estats estacionaris són estables.

**Exercici 14. Dues economies.** Considera el model de Solow i Swan sense creixement de la població i sense progrés tecnològic. Hi ha dues economies. En cadascuna d'elles la taxa d'estalvi és  $1/2$  i la funció de producció agregada és  $Y = K^{1/2} \cdot L^{1/2}$ . En una de les economies la taxa depreciació  $\delta$  del capital per càpita és  $\delta = 1/2$ . En l'altra és  $\delta = 1/4$ . El volum de població és el mateix a cada economia. (i) Calcula els estats estacionaris de cada economia. (ii) Imagina que les dues economies s'integren i en formen una de sola, de manera que ara a l'economia hi ha dos grups d'individus, cada grup corresponent a una de les economies originàries. Assumint que el capital que acumula cada grup es deprecia a la mateixa taxa de l'economia que formava inicialment el grup, determina els estats estacionaris de la nova economia i compara'ls amb els de les dues economies originals.

**Exercici 15. Con crecimiento de la población.** La tasa de ahorro es  $1/2$ . La función de producción agregada es  $Y_t = 2 \cdot K_t^{1/2} \cdot L_t^{1/2}$ . La tasa de depreciación es  $1/12$ . La tasa de crecimiento de la población es  $1/4$  si el capital per cápita es inferior a 24 y es  $1/12$  en caso contrario.

- (i) Calcula todos los estados estacionarios, represéntalos gráficamente y explica cuáles de ellos son estables y cuáles no. Interpreta los resultados.
- (ii) Responde (i) y determina la tasa de ahorro que satisface la regla de oro.
- (iii) Responde (ii) si la función de producción agregada es  $Y_t = 3 \cdot K_t^{1/3} \cdot L_t^{2/3}$ .

**Exercici 16. Con función de producción no convencional.** La tasa de ahorro es  $1/4$ . La función de producción per cápita es  $y_t = k_t^{1/2}$  si  $k_t \leq 16$ ; es  $y_t = 4 + (k_t - 16)^{1/2}$  si  $16 < k_t < 100$ ; y es  $y_t = 10 + (k_t - 100)^{1/2}$  si  $k_t \geq 100$ . La tasa de depreciación es un valor fijo  $\delta$ , entre 0 y 1. (i) Calcula todos los estados estacionarios, represéntalos gráficamente y explica cuáles de ellos son estables y cuáles no. (ii) Interpreta los resultados.

**Exercici 17. Con función de ahorro no convencional.** La función de depreciación es  $d_t = 1 + \delta \cdot k_t$ , donde  $\delta$  es un valor fijo entre 0 y 1. La función de producción agregada es  $Y_t = 2 \cdot K_t^{1/2} \cdot L_t^{1/2}$ . La tasa de ahorro es  $1/2$  si el capital per cápita no es superior a 2 y es  $1/k$  en caso contrario. (i) Calcula todos los estados estacionarios, represéntalos gráficamente y explica cuáles de ellos son estables y cuáles no. (ii) ¿Tiene sentido la función de depreciación?

**Exercici 18. Regla de oro.** La función de producción per cápita es  $y_t = k_t^\alpha$ , donde  $0 < \alpha < 1$ . Con tasa  $\delta$  de depreciación y tasa  $n$  de crecimiento de la población, demuestra que la tasa de ahorro que satisface la regla de oro es  $\alpha$ .

**Exercici 19. Regla de oro.** La tasa de ahorro es igual a la tasa de depreciación. La función de producción per cápita es  $y = k^{1/2}$ . Representa gráficamente el modelo, y los estados estacionarios, y explica si la tasa de ahorro de la economía es o no la tasa de ahorro que satisface la regla de oro. Si no lo es, ¿cuál de las dos es mayor?

**Exercici 20. Con función de producción atípica.** La función de producción agregada es  $Y_t = K_t^2/L_t$  cuando  $K_t/L_t < 2$  y es  $Y_t = 4 \cdot L_t + K_t^{1/2} \cdot L_t^{1/2}$  en caso contrario. La tasa de ahorro es  $1/2$ . La tasa de depreciación es  $1/4$ . Calcula todos los estados estacionarios, represéntalos gráficamente y explica cuáles de ellos son estables y cuáles no.

**Exercici 21. Dos economías.** Hay dos economías, E1 y E2. En E1, la función de producción agregada es  $Y_t = K_t^{1/2} \cdot L_t^{1/2}$ , la tasa de depreciación es  $1/10$  y la tasa de ahorro es  $1/2$ . En E2, la función de producción agregada es  $Y_t = K_t^{1/3} \cdot L_t^{2/3}$ , la tasa de depreciación es  $1/10$  y la tasa de ahorro es  $1/4$ .

- (i) Calcula todos los estados estacionarios, represéntalos gráficamente y explica cuáles de ellos son estables y cuáles no.
- (ii) Responde al apartado (i) si las dos economías se fusionan.

**Exercici 22. Con funciones de ahorro y depreciación atípicas.** La función de producción agregada es  $Y_t = 2 \cdot K_t^{1/2} \cdot L_t^{1/2}$ . La tasa de ahorro es  $1/2$  para valores del capital per cápita menores de 10 o mayores de 24; la tasa de ahorro es  $1/4$  para valores del capital per cápita entre 10 y 24. La función de depreciación toma el valor constante 2 para valores del capital per cápita inferiores a 20 y viene definida por la expresión  $d_t = 2 + (k_t - 20)/4$  en caso contrario.

- (i) Calcula todos los estados estacionarios, represéntalos gráficamente y explica cuáles de ellos son estables y cuáles no.
- (ii) Responde al apartado (i) si la tasa de ahorro es  $1/4$  para valores del capital per cápita menores de 10 o mayores de 24 y es  $1/2$  para valores del capital per cápita entre 10 y 24.

**Exercici 23. Rompecabezas.** El capital per cápita que satisface la regla de oro es 10. La tasa de depreciación es  $1/10$ . La función de producción per cápita es  $y_t = k_t^\alpha$ , donde  $0 < \alpha < 1$ . Calcula  $\alpha$ , la tasa de ahorro y la tasa de ahorro que satisface la regla de oro.

**Exercici 24. Solow y Swan creativo.** Partiendo del de Solow y Swan, construye y analiza un modelo en el que la función de producción sea  $Y_t = A_t \cdot K_t^\alpha \cdot L_t^{1-\alpha}$  y en donde el estado de la tecnología  $A_t$  progresa según reglas análogas a las de la acumulación de capital, esto es, una fracción fija de la producción se destina a mejorar el estado de la tecnología al tiempo que este estado decae a una tasa constante dada.

**Exercici 25. Solow y Swan extravagante.** Propón ecuaciones para la función de ahorro y la función de depreciación que tengan la representación gráfica de la derecha. Calcula los valores numéricos de todos los estados estacionarios correspondientes a las ecuaciones que has propuesto y explica, con ayuda de la gráfica, si son estables.

