2. An overlapping generations model with private and public lending

1. Taxes

This and the following extensions of the OLG model with only private lending introduce an immortal agent: the government. Suppose a government is created that merely taxes endowments.

Definition 1.1. A tax scheme τ_t^i for individual i of generation t is a pair $\left(\tau_t^i(t), \tau_t^i(t+1)\right)$, where $\tau_t^i(s)$ is the tax that $i \in N(t)$ pays (or receives) in period $s \in \{t, t+1\}$.

Remark 1.2. A negative tax will be called "transfer".

Definition 1.3. The government budget constraint when taxes are just paid out as transfers states that, for all $t \geq 1$,

$$\sum_{i \in N(t)} \tau_t^i(t) + \sum_{i \in N(t-1)} \tau_{t-1}^i(t) = 0.$$

To compute the general competitive equilibrium (GCE), consider the no tax case and replace $w_t^i(s)$ with $w_t^i(s) - \tau_t^i(s)$. The only additional condition to calculate the GCE is the government budget constraint. In particular, the new lifetime budget constraint of consumer i is

$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = w_t^i(t) - \tau_t^i(t) + \frac{w_t^i(t+1) - \tau_t^i(t+1)}{R(t)}. \quad (1)$$

Define the savings $s^i(t)$ of (young) individual i as the part of i's disposable endowment $w_t^i(t) - \tau_t^i(t)$ that is not consumed. That is,

$$s^i(t) = w_t^i(t) - \tau_t^i(t) - c_t^i(t). \quad (2)$$

Remark 1.4. With savings defined as in (2), the equilibrium interest rate $R(t)$ in t is obtained (as in the no tax case) from the condition $\sum_{i \in N(t)} s^i(t) = 0$. If $s^i(t)$ is still defined as in the no tax case (that is, $s^i(t) = w_t^i(t) - c_t^i(t)$), then the equilibrium condition is $\sum_{i \in N(t)} s^i(t) = \sum_{i \in N(t)} \tau_t^i(t)$.

2. Government bonds

Definition 2.1. A (one-period) bond is a (safe) promise (by the government) of delivering 1 unit of the good at $t+1$ in exchange for a (competitive) price $p(t) < 1$ paid to the government in period t.

Hence, $p(t)$ is the price of the bond (when issued in period t) and the face (or nominal) value of the bond is 1. This way of defining bonds means that they are issued at a discount (selling price smaller than its face value).

Definition 2.2. The (implicit) rate of return of the bond is $\frac{1-p(t)}{p(t)}$. The gross rate of return of the bond is then $1+\frac{1-p(t)}{p(t)}=\frac{1}{p(t)}$: the investor gets 1 in $t+1$ by investing $p(t)$ in t.

Assume now that the government can issue bonds and collect taxes. Though only one agent supplies bonds in the bond market (the government), it will be assumed that the market is competitive: supply of bonds in t and demand for bonds in t determine the price $p(t)$ of bonds in t.

Definition 2.3. For $t \geq 1$, let $B(t)$ stand for the total number of bonds that the government issues in period t. Given that the face value of each bond is 1, $B(t)$ also represents the debt that the government must pay in period $t+1$.

Definition 2.4. The government budget constraint in period t states that

$$\underbrace{B(t-1)}_{\textit{debt to be paid}} = \underbrace{\textstyle\sum_{i\in N(t)} \tau_t^i(t)}_{\textit{taxes on the young}} + \underbrace{\textstyle\sum_{i\in N(t-1)} \tau_{t-1}^i(t)}_{\textit{taxes on the old}} + \underbrace{p(t)B(t)}_{\textit{new bonds}}.$$

The constraint shows the three ways of redeeming in t bonds issued in $t-1$: tax the young in t; tax the old in t; and issue new bonds in t (now $\sum_{i\in N(t)} \tau_t^i(t) + \sum_{i\in N(t-1)} \tau_{t-1}^i(t)$ need not be zero).

Since the old individuals never lend, the government can only borrow from (sell bonds to) the young individuals. Hence, only young individuals will buy bonds. In thisi case, a young individual i of generation t faces the budget constraint

$$c_t^i(t) + l^i(t) + \tau_t^i(t) + p(t)b^i(t) = w_t^i(t).$$

This says that there are four possible uses for i's wealth $w_t^i(t)$: it can be consumed, lent in the private loan market, given to the government (in the form of taxes), or lent to the government (by purchasing the amount $b^i(t)$ of bonds).

The budget constraint of an old individual i of generation t is

$$c_t^i(t+1) + \tau_t^i(t+1) = w_t^i(t+1) + R(t)l^i(t) + b^i(t).$$

Putting together the above two constraints (solve for $l^i(t)$ in the first equation and insert the result in the second), individual i's lifetime budget constraint turns out to be

$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = w_t^i(t) - \tau_t^i(t) + \frac{w_t^i(t+1) - \tau_t^i(t+1)}{R(t)} - b^i(t)\left[p(t) - \frac{1}{R(t)}\right]. \qquad (3)$$

Proposition 2.5. *In a general competitive equilibrium, and assuming arbitrage,* $p(t) = \frac{1}{R(t)}$.

Proof. Consider an individual that would like to save $p(t)$ units of the good in period t. The individual has two options.

• **Option 1.** To become a lender in the government bond market. Since $p(t)$ is the price of one bond in t, the outcome (profit) of this saving decision is 1 unit of the good in the next period $t+1$.

• **Option 2.** To become a lender in the private loan market. By lending $p(t)$ units of the good in the private loan market (assumed competitive), and given the (gross) interest rate $R(t)$ in t, it follows that the individual obtains $p(t)R(t)$ units of the good in the next period $t+1$.

By arbitrage in the two markets, both options should yield the same result; that is, $1 = p(t)R(t)$: lending to the government must generate the same profit as lending to individuals.

For if $1 > p(t)R(t)$, then public lending would be more profitable than private lending. By borrowing $p(t)$ in the private loan market to purchase one bond, in $t+1$ the bond pays 1, whereas the refund of the loan requires $p(t)R(t)$. A sure profit of $1 - p(t)R(t)$ is made. But in a GCE sure profits cannot arise. A growing demand for both loans and bonds cause a rise in $R(t)$ and $p(t)$.

Similarly, arbitrage opportunities also occur if $1 < p(t)R(t)$ (public lending is less profitable than private lending). Consequently, for both markets to exist (for lenders to be willing to participate in both markets) the corresponding returns should be the same: $1 = p(t)R(t)$, which implies $\frac{1}{p(t)} = R(t)$, where $\frac{1}{p(t)}$ represents the (gross) interest rate of the bond and $R(t)$ is the (gross) interest rate of a (private) loan. Clearly, $\frac{1}{p(t)} = R(t)$ entails $p(t) - \frac{1}{R(t)}$. The term $b^i(t)\left[p(t) - \frac{1}{R(t)}\right]$ in (3) is, as a result, zero. To sum up, the lifetime budget constraint of each young individual i is identical to the one from the no bond case: equation (1).■

3. General competitive equilibrium with bonds

Recall that the aggregate savings function S_t, derived from the maximization of the consumers' utility function subject to their lifetime budget constraints, was a function of the interest rate $R(t)$ and the consumers' endowments. To simplify notation, and given that endowments are held fixed, it will be only emphasized that savings depend on $R(t)$ by writing $S_t(R(t))$.

Proposition 3.1. *In a general competitive equilibrium of the economy with public and private lending, the equilibrium interest rate* $R(t)$ *is determined by* (4), *if savings are defined as in* (2).

$$S_t(R(t)) = p(t)B(t) \tag{4}$$

Proof. There are now three markets in the economy: the market for the (consumption of the) good; the (private) loan market; and the (government) bond market. In a general competitive equilibrium all three markets must clear (must be in equilibrium). By Walras' law equilibrium in two markets guarantees equilibrium in the third one.
The summation of the budget constraints of all the young individuals in period t yields

$$\sum_{i\in N(t)} c_t^i(t) + \sum_{i\in N(t)} l^i(t) + \sum_{i\in N(t)} \tau_t^i(t) + p(t)\sum_{i\in N(t)} b^i(t) = \sum_{i\in N(t)} w_t^i(t).$$

Equilibrium in the loan market requires $\sum_{i\in N(t)} l^i(t) = 0$. Rearranging,

$$\sum_{i\in N(t)}\left[w_t^i(t) - c_t^i(t) - \tau_t^i(t)\right] = p(t)\sum_{i\in N(t)} b^i(t).$$

By (2), $w_t^i(t) - c_t^i(t) - \tau_t^i(t) = s^i(t)$. Consequently, $\sum_{i\in N(t)} s^i(t) = p(t)\sum_{i\in N(t)} b^i(t)$. On the other hand, equilibrium in the bond market amounts to $\sum_{i\in N(t)} b^i(t) = B(t)$; that is, demand for bonds equals supply of bonds. All in all, $\sum_{i\in N(t)} s^i(t) = p(t)B(t)$.■

Remark 3.2. If all the budget constraints are assumed to hold and all individuals maximize their utility functions subject to the corresponding budget constraints, then that (4) is satisfied implies that all markets are in equilibrium.

Remark 3.3. By Proposition 3.1 and Remark 3.2, (4) suffices to calculate the equilibrium interest rate and, from that value, the rest of variables of a general competitive equilibrium.

The general equilibrium condition (4) holds that total private (net) savings by the young in t must equal the total value of the goverment debt in t. Insofar as $R(t) = 1/p(t)$, (4) can be equivalently expressed as

$$S_t(R(t)) = \frac{B(t)}{R(t)},$$

so aggregate savings in t equal the present value of the debt $B(t)$ in $t+1$. Remind that the debt in $t+1$ corresponds to the amount of bonds issued in t: one bond issued in t involves a debt of one unit of the good in $t+1$.

4. An example of general equilibrium with bonds and taxes

Example 4.1. For all t and i, $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$, $N(t) = 100$, individuals in each generation are numbered from 1 to 100, $w_t^i = (2,0)$ if i is odd, and $w_t^i = (1,1)$ if i is even. The government wishes to borrow 25 units of the good in $t = 1$, transfer them to the old in $t = 1$, and pay off the debt by taxing the young individuals in $t = 2$: each such individual pays the amount τ in taxes. The aim is to find $R(1)$, $B(1)$, $R(2)$, $R(3)$, and τ.

When no tax is paid in t, the savings function is $s^i(t) = 1$ if i is odd and $s^i(t) = \frac{1}{2} - \frac{1}{2R(t)}$ if i is even. The aggregate savings function is then $S_t = 50(1) + 50\left(\frac{1}{2} - \frac{1}{2R(t)}\right) = 75 - \frac{25}{R(t)}$.

General competitive equilibrium in t requires $S_t = p(t)B(t)$. For $t = 1$, $p(1)B(1) = 25$. This is because the government would like to raise 25 units of the good by selling the amount $B(1)$ of bonds. Therefore, in equilibrium in period 1, $S_1 = 25$. Since $S_1 = 75 - \frac{25}{R(1)}$, it follows that $75 - \frac{25}{R(t)} = 25$ and, hence, $R(1) = \frac{1}{2}$.

Savings in $t = 1$ are $s^i(1) = 1$ for i odd and $s^i(1) = -\frac{1}{2}$ for i even. In words: in period 1 odd-numbered individuals lend 50 units in total, while even-numbered individuals borrow 25 units in total. The difference (25 units) is what the government borrows.

Using $S_1 = B(1)/R(1)$, with $S_1 = 25$ and $R(1) = 1/2$, the conclusion is that $B(1) = 12.5$. This is the amount of bonds issued in $t = 1$ and the total amount of taxes that the young individuals in $t = 2$ will have to pay. Thus, $\tau = 0.125$.

Consider now period 2. As young individuals must pay the tax, the lifetime budget constraint of a young individual i is $c_2^i(2) + \frac{c_2^i(3)}{R(2)} = 2 - \tau$ if i is odd and $c_2^i(2) + \frac{c_2^i(3)}{R(2)} = 1 - \tau + \frac{1}{R(2)}$ if i is even.

For i odd the demand function for the good (when i is young) is $c_2^i(2) = 1 - \frac{\tau}{2}$. As a result, his savings function is $s^i(2) = 2 - \tau - c_2^i(2) = 1 - \frac{\tau}{2}$. In consequence, i pays the tax τ by reducing consumption and savings in the same amount: $\frac{\tau}{2}$. It can be interpreted that a half of the tax is financed by reducing consumption and the other half by reducing savings.

For i even the demand function for the good (when i is young) is $c_2^i(2) = \frac{1}{2} + \frac{1}{2R(2)} - \frac{\tau}{2}$. In view of this, his savings function is $s^i(2) = 1 - \tau - c_2^i(2) = \frac{1}{2} - \frac{\tau}{2} - \frac{1}{2R(2)}$. Just like an odd i, an even i pays the tax τ by reducing consumption and savings in the same amount: $\frac{\tau}{2}$.

The aggregate savings function is $S_2 = 50\left(1 - \frac{\tau}{2}\right) + 50\left(\frac{1}{2} - \frac{\tau}{2} - \frac{1}{2R(2)}\right) = 75 - 50\tau - \frac{25}{R(2)}$. Given that $\tau = 0.125$, $50\tau = 6.25$. Summing up, $S_2 = 58.75 - \frac{25}{R(2)}$.

Presuming that $B(2) = 0$ (the government has no need to borrow in period 2), the equilibrium condition turns out to be $S_2 = 0$. Therefore, $R(2) = \frac{25}{58.7\ 5} = \frac{1}{2.35} \approx 0.4255$.

As regards $R(3)$, it is as if the government disappeared in period 3: no tax and no bond market. The aggregate savings function is as in period 1: $S_3 = 75 - \frac{25}{R(3)}$. With the equilibrium condition now being $S_3 = 0$, it follows that $R(2) = \frac{1}{3}$. The same result holds for the rest of periods, as long as the government does not issue more debt or does not introduce taxes.

Hence, if the situation in period 3 is supposed to represent the initial situation in period 0, Example 4.1 suggests that the interest rate may go up because of a rise in the government debt or an increase in the taxes paid by individuals.

As rising taxes is not a popular economic policy measure, the following section considers the possibility that the government pays off bonds by, instead of rising taxes, issuing more bonds.

5. Rolling over debt

Definition 5.1. A goverment rolls over debt when debt is paid off with new debt.

Example 5.2. In Example 4.1 the young in $t = 2$ are not taxed: new bonds are issued in $t = 2$ to pay off the amount $B(1) = 12.5$ of bonds issued in $t = 1$.

Now, in equilibrium, $S_2 = B(2)/R(2)$ and $S_2 = B(1)$. Therefore, $R(2) = 0.4$ and $B(2) = 5$. If the same policy is followed in period $t = 3$, $S_3 = B(3)/R(3)$ and $S_3 = B(2)$. Accordingly, $R(3) = 0.35$ and $B(3) = 1.78$. The acumulation of bonds and the dynamics of the interest rate are determined by the formulae

$$R(t) = \frac{25}{75 - B(t-1)} \quad \text{and} \quad B(t) = \frac{25B(t-1)}{75 - B(t-1)}. \tag{5}$$

Definition 5.3. A steady state is one in which equilibrium variables take the same value in every t.

In particular, a steady state would require $B(t-1) = B(t)$. This occurs in two cases: (i) $B = 50$ and $R = 1$; and (ii) $B = 0$ and $R = 1/3$ ($R = 1/3$ is the equilibrium rate in absence of government intervention).

The formulae in (5) hold when the government borrows initially at most 50, so $S(1) \leq 50$. If $S(1) < 50$, $B(t)$ goes to 0 and $R(t)$ converges to $1/3$; see Table 1 and Fig. 2. If $S(1) > 50$, borrowing becomes unfeasible for some t: a bubble eventually arises, which means that the price of the bonds follows an unsustainable path; see Table 3 and Fig. 4.

According to the values in Fig. 3, in period $t = 14$ the goverment asks for more units of the good (1151) than are available in that period (200). To make this part of an equilibrium, a negative gross interest rate R would be required. Yet a negative R cannot arise in equilibrium, because $R < 0$ means that, after lending one unit of the good in t, rather than receive good, you must still pay more good in $t + 1$. A negative R is like a tax on lending. When $R < 0$ it is plain that consuming is better than lending. In fact, the following cases may arise:

- if $r(t) > 0$ ($R(t) > 1$), then, by lending L at t, you get more than L at $t + 1$;

• if $-1 \leq r(t) \leq 0$ ($0 \leq R(t) \leq 1$), by lending L at t, you get less than L at $t+1$;

• if $r(t) < -1$ ($R(t) < 0$), by lending L at t, you have to pay at $t+1$. In this case, in equilibrium, no one lends: the sacrifice of current consumption that represents lending yields no future benefit, so a utility maximizer consumer will consume everything that is available (actually, everybody would like to borrow).

t	$S(t)$	$R(t)$	$B(t)$	% change in $B(t)$
1	49.99	0.9996	49.97001	–
2	49.97001	0.998802	49.91014	−0.11981
3	49.91014	0.996419	49.7314	−0.35814
4	49.7314	0.98937	49.20276	−1.06299
5	49.20276	0.969096	47.68218	−3.09042
6	47.68218	0.915154	43.63652	−8.48464
7	43.63652	0.797105	34.78291	−20.2895
8	34.78291	0.621626	21.62197	−37.8374
9	21.62197	0.468358	10.12681	−53.1642
10	10.12681	0.385367	3.902542	−61.4633
11	3.902542	0.35163	1.372251	−64.837
12	1.372251	0.339546	0.465942	−66.0454
13	0.465942	0.335417	0.156285	−66.4583
14	0.156285	0.334029	0.052204	−66.5971
15	0.052204	0.333566	0.017413	−66.6434

Table 1. Dynamics when the government borrows at most 50 in $t = 1$

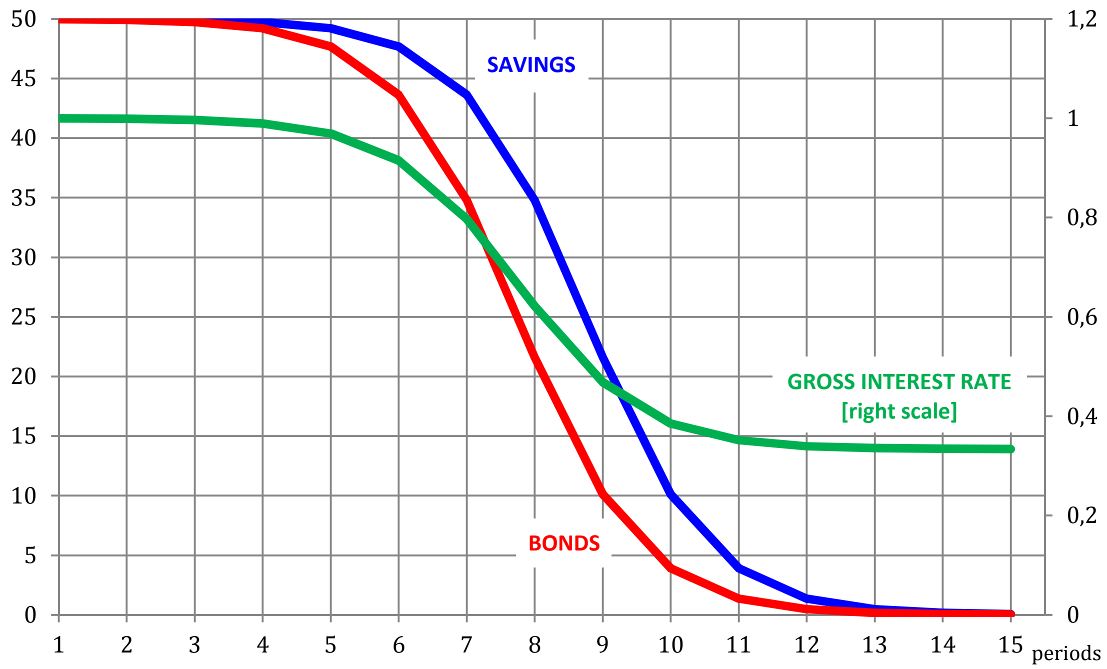

Fig. 2. Dynamics when the government borrows at most 50 in $t = 1$

To put it in a nutshell, in period 14 the debt bubble bursts. Fig. 4 depicts the data from Table 3. The analysis developed above suggests Proposition 5.4, a result which is not formally proved and is left as an informal, intuitive claim.

t	$S(t)$	$R(t)$	$B(t)$	% change in $B(t)$
1	50.00001	1	50.00003	–
2	50.00003	1.000001	50.00009	0.00012
3	50.00009	1.000004	50.00027	0.00036
4	50.00027	1.000011	50.00081	0.00108
5	50.00081	1.000032	50.00243	0.00324
6	50.00243	1.000097	50.00729	0.009721
7	50.00729	1.000292	50.02188	0.029173
8	50.02188	1.000876	50.0657	0.087595
9	50.0657	1.002635	50.19761	0.263477
10	50.19761	1.007967	50.59755	0.796729
11	50.59755	1.024487	51.83654	2.448716
12	51.83654	1.079286	55.94645	7.928594
13	55.94645	1.312091	73.40684	31.20912
14	73.40684	15.69205	1151.904	1469.205
15	1151.904	–0.02321	–26.7411	–102.321

Table 3. Dynamics when the government borrows more than 50 in $t = 1$

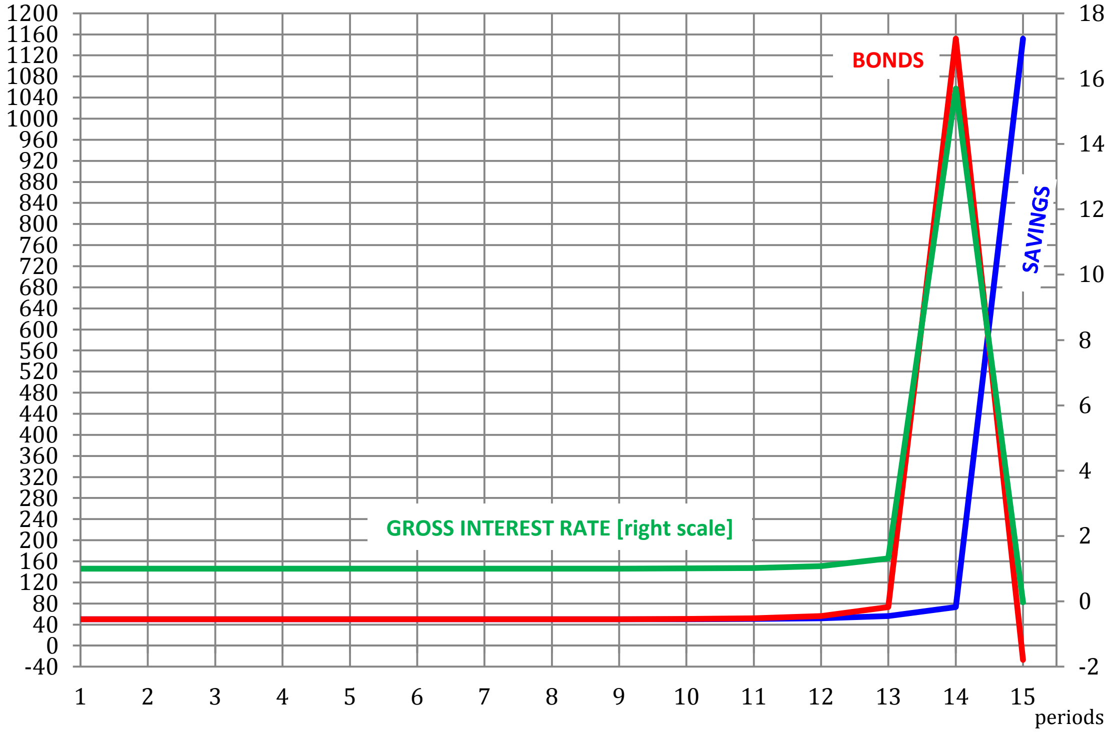

Fig. 4. Dynamics when the government borrows more than 50 in $t = 1$

Proposition (informal) 5.4. *To sustain a growing government debt, population or endowments must grow.*

Example 5.5. Consider the economy from Example 5.2 without taxes and where endowments double each period.

Then $S_t = \left(75 - \frac{25}{R(t)}\right) 2^{t-1}$, $R(t) = \frac{25}{75 - \frac{B(t-1)}{2^{t-1}}}$, and $B(t) = \frac{25B(t-1)}{75 - \frac{B(t-1)}{2^{t-1}}}$. As a consequence, the government can now borrow initially 62 but not 63; see Table 5.

t	$S(t)$	$R(t)$	$B(t)$
1	62	1,92	119,2
2	119,2	1,62	193,7
3	193,7	0,94	182,3
4	182,3	0,47	87,3
5	87,3	0,35	31,3
6	31,3	0,337	10,6
7	10,6	0,334	3,5

$S(t)$	$R(t)$	$B(t)$
63	2,08	131,2
131,25	2,66	350
350	−2	−700
−700	0,15	−107,6
−107,6	0,3	−32,9
−32,9	0,32	−10,8
−10,8	0,33	−3,6

Table 5. Dynamics when the government borrows 62 or 63 in $t = 1$ in the economy of Example 5.2 (the yellow colour indicates an impossible situation)

6. On the equivalence between bonds and taxes

Apparently, financing government debt issuing new debt does not seem to be the same thing as financing by increasing taxes. Individuals may dislike having to pay more taxes because that (as illustrated in Example 4.1) will shrink their consumption possibilities. But (as suggested by also Example 4.1) replacing taxes by bond issue will lead to an increase in the interest rate, which harms borrowers. Moreover, the strategy of rolling over debt may lead to explosive situations in which the (private) loan market collapses (so both lenders and borrowers end up being worse off).

Proposition 6.1 below states that, at least with respect to equilibrium consumption allocations, what can be achieved through bonds can be replicated using taxes.

Proposition 6.1. *Let C be an equilibrium consumption allocation with bonds. Then, for some tax-transfer scheme (without bonds) that balances the government's budget in each period t (total taxes in t equal total transfers in t), C is also an equilibrium consumption allocation.*

Proof. With bonds, the equilibrium interest rate $\hat{R}(t)$ in t solves $S_t\left(\hat{R}(t)\right) = B(t)/\hat{R}(t)$. Given $\hat{R}(t)$ and the bond holdings $b^i(t)$, the same equilibrium consumption allocation can be obtained with taxes (but without bonds) by setting $\tau_t^i(t) = b^i(t)$ and $\tau_t^i(t+1) = -\hat{R}(t) \cdot b^i(t)$.■

Remark 6.2. The so-called Ricardian equivalence proposition (attributed to David Ricardo, 1772-1823) is the result according to which the method of financing government spending (bonds or taxes) does not affect the consumers' decisions. The result relies on the presumption that consumers internalize the government's budget constraint when making consumption decisions. Proposition 6.3 next is a rough way of formulating this equivalence.

Proposition (informal) 6.3. *Consumption allocations and interest rates do not change if the government borrows now and "appropriately" taxes later instead of just taxing now.*

The proof of Proposition 6.3 would amount to showing that moving from one policy (borrow now and tax later) to the other (tax now) does not alter the consumer's present value of endowments.

The equivalence could fail if, for instance, one policy is to borrow from generation 1 and tax generation 2 while the other is to tax generation 1. As different generations would be involved, the budget constraints of some individuals might be different, in which case their consumption decisions could be altered. Example 6.4 illustrates Proposition 6.3.

Example 6.4. Members of each generation t are all identical, with $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$. This implies that there is no private borrowing. In view of this, the price $p(t)$ of bonds can be normalized to 1 in each t. There are two policies.

- P1: set tax $\tau_t^i(t) = m$.

- P2: borrow m in t from generation t and tax in $t+1$ generation t to pay off the bonds issued in t.

Under P1, the consumption basket is $c_t^i = \left(w_t^i(t) - m, w_t^i(t+1)\right)$. Under P2, $c_t^i = \left(w_t^i(t) - b^i(t), w_t^i(t+1) + R(t)b^i(t) - \tau_t^i(t+1)\right)$. Since taxes must only pay off the bonds, $\tau_t^i(t+1) = R(t)b^i(t)$. But $b^i(t) = m$, so $\tau_t^i(t+1)$ has present value m ($m = \tau_t^i(t+1)/R(t)$). This means that the present value of i's tax liability is not altered: it is m (in t) under P1 and $m \cdot R(t)$ (in $t+1$) under P2. <u>The consumption basket is the same under the two policies</u>. Moreover, in equilibrium, $R(t) = MRS_t^i$. As $MRS_t^i = \frac{c_t^i(t+1)}{c_t^i(t)}$, the MRS does not change. Accordingly, the interest rate is the same under both policies. Summing up, P1 and P2 are equivalent policies.

7. Pensions

Consider an economy in which individuals initially can only consume or lend/borrow in the loan market. A government that plans to introduce a pension system has two basic options.

Definition 7.1. In a <u>fully funded pension system</u> the government taxes the young in t, lends the revenues, and pays out the proceeds to the old in the next period $t+1$ as a pension.

Definition 7.2. In an unfunded or <u>pay-as-you-go pension (PAYGO) system</u> the pension $p(t)$ to the old in period t are paid out from current tax receipts $\tau(t)$ on the young.

Under fully funded pensions, a young individual pays his own pension through the government: the government takes savings from young individuals, invests them, and, when the individuals become old, are paid the pension from the revenues of the investment. This procedure suggests the following question: are the young people forced to save more than they wish? In fact, when young, i's budget constraint is

$$c_t^i(t) + l^i(t) + \tau(t) = w_t^i(t);$$

when old, it is

$$c_t^i(t+1) = w_t^i(t+1) + R(t) \cdot \left[l^i(t) + \tau(t)\right]$$

where l^i represents what i voluntary saves and τ is the tax paid. The government invests τ to get the market return R and, in the next period, transfer the proceeds $R \cdot \tau$ to the individual.

Inspection of the budget constraints indicates that <u>the pension has no effect</u> on the savings decision since budget constraints formally coincide with the constraints without the pension: $l^i + \tau$ simply replaces l^i. Voluntary savings are cut to pay taxes so that income remains the same. More specifically, if i chooses to save l^i without pension, then, under the fully-funded pension system with tax τ, the individual saves $l^i - \tau$. Even though the government manages τ, in the end, the individual receives the same proceeds as if he managed τ.

The interesting case is then the PAYGO system: the pension $p(t)$ assigned to the old in t comes from the taxes $\tau(t)$ currently paid by a different generation, the young people. To illustrate the implications of the system, suppose population grows at rate n. Assuming that the government chooses to <u>balance the budget</u>, the government budget constraint in period t is

$$\tau(t) \cdot N(t) = p(t) \cdot N(t-1).$$

In words, the total amount $\tau(t) \cdot N(t)$ of tax receipts in t from the generation born in t must equal the total amount $p(t) \cdot N(t-1)$ of pensions paid in t to the generation that is old in t (hence, the generation born in $t-1$). As population grows at a constant rate n,

$$\tau(t) \cdot (1+n) \cdot N(t-1) = p(t) \cdot N(t-1)$$

and, therefore,

$$\tau(t) \cdot (1+n) = p(t).$$

When young, i's budget constraint is

$$c_t^i(t) + l^i(t) + \tau(t) = w_t^i(t);$$

when old, it is

$$c_t^i(t+1) = w_t^i(t+1) + R(t) \cdot l^i(t) + p(t) = w_t^i(t+1) + R(t) \cdot l^i(t) + \tau(t) \cdot (1+n).$$

His lifetime budget constraint is

$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = w_t^i(t) + \frac{w_t^i(t+1)}{R(t)} + \tau(t) \cdot \left(\frac{1+n}{1+r(t)} - 1\right).$$

Without the pension, the term $\tau(t) \cdot \left(\frac{n-r}{1+r(t)}\right)$ is missing. If $n > r(t)$, then the budget set with the pension is larger, so <u>a more preferred consumption basket is feasible</u>. If $n < r(t)$, then the budget set with the pension is smaller. As the welfare maximizing basket without the pension is not feasible now, <u>the pension reduces the young's welfare</u>. The whole system is like a <u>pyramid scheme</u>: a club is created so that new members must pay to existing members and will be paid by future members. Loosely speaking, a new member pays r, whereas the availability of future members is determined by n. As long as the pool of potential members is larger than the club of existing members, it is profitable to join the club.

In an economy with an expanding population, young people could finance the pensions of the current old. The precise condition is actually $n > r$, for it is not 1 that has to be transferred to an old individual, but $1 + r$: for every unit of good that the old did contribute as young, he is entitled to receive $1 + r$. Hence, if each young individual pays 1, a total of $1 + n$ individuals is required.

Example 7.3. Suppose $r = 1/2$ and that there are 100 old individuals. If the taxes each old individual paid when young is normalized to 1, then the collective of old individuals is entitled to receive pensions worth $100 \cdot \left(1 + \frac{1}{2}\right) = 150$. If the number of currently young individuals remained at 100 (indicating that $n = 0$), then, <u>if taxes do not change</u>, the government can only collect 100 from the young people. This amount is insufficient to cover the need of 150. Accordingly, n must at least be such that $100 \cdot (1 + n) = 150$. That is, $n = 1/2$ is the smallest value that makes the system viable when $r = 1/2$ and taxes do not change.

If taxes can change, then to sustain a PAYGO system when $n < r$, the young must be over-taxed with respect to previous generations. In the end, what matters is that, collectivelly, the young can finance the old. This can be done by increasing the number of tax-payers and/or by increasing what each tax-payer contributes.

8. Exercicis

Exercici 1. Equilibri amb bons. La funció d'utilitat de cada consumidor i és $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$. Cada generació està formada per 100 membres, 80 amb dotació $(1, 0)$ i els altres 20 amb dotació $(2, 0)$. El govern pretén aplegar 10 unitats amb l'emissió de bons amb venciment d'un període. Al venciment, els bons es paguen amb l'emissió de més bons, també amb venciment d'un període. I així successivament.

(i) Calcula la taxa d'interès d'equilibri, el preu dels bons i la quantitat de bons emesa en els períodes 1, 2 i 3.

(ii) Respon l'apartat (i) si la dotació dels individus del grup de 20 és $(2, 1)$ en comptes de $(2, 0)$.

(iii) En el cas (ii), troba un import inicial a aplegar que provoqui que el refinançament continuat del deute faci que el volum de bons emès cada període sigui el mateix.

(iv) En el cas (ii), indica un import inicial que faci eventualment insostenible el refinançament del deute.

(v) Respon l'apartat (i) amb les dades del (ii) si, en el període 2, traspassen la meitat dels consumidors joves amb dotació $(1, 0)$

Exercici 2. Condició d'equilibri general. Demostra la Remarca 3.2.

Exercici 3. Funcions d'estalvi. Calcula les funcions d'estalvi agregat de l'Exemple 5.2 corresponents als períodes 1, 2 i 3.

Exercici 4. Equivalència de bons i imposts. Considera l'economia tal que, per a tot t i i, $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$, $N(t) = 100$, els consumidors de cada generació estan numerats de l'1 al 100, $w_t^i = (2,0)$ si i és senar i $w_t^i = (1,1)$ si i és parell. El govern vol manllevar 25 unitats del bé en el període 1 mitjançant la venda de bons i refinança el deute generat pels bons cada període emetent més bons. Troba l'esquema d'imposts i transferències que genera la mateixa assignació de consum d'equilibri que la política de refinançament del deute amb més bons.

Exercici 5. Finançament d'un bé públic. Cada generació té 100 membres: 50 d'ells ("els pobres") amb dotació $(1,0)$ i els altres 50 ("els rics") amb dotació $(4,1)$. Els consumidors, rics o pobres, empren la dotació en consum c, préstecs (privats) l i contribucions (voluntàries) e a un bé públic.

El bé públic només beneficia als consumidors joves (pots suggerir algun exemple real d'aquesta situació?). Per consegüent, la gent gran no contribueix al bé públic. La funció d'utilitat de cada consumidor (jove) i és $u_t^i = c_t^i(t) \cdot c_t^i(t+1) \cdot [1 + g(\sum_{j \in N(t)} e^j)]$, on e^j és la contribució del consumidor jove j (e^j no pot ser negativa ni superior a la dotació que j té de jove) i on g és una mena de funció de producció del bé públic: el total de contribucions $\sum_{j \in N(t)} e^j$ genera el volum $g\left(\sum_{j \in N(t)} e^j\right)$ de bé públic. Pot interpretar-se que cada unitat de bé públic fa més útil el consum privat del bé. Per a simplificar, $g\left(\sum_{j \in N(t)} e^j\right) = \sum_{j \in N(t)} e^j$.

(i) Determina quina és la contribució e^P a finançar el bé públic que, en l'equilibri general, fa un consumidor pobre i quina és la contribució e^R que fa un consumidor ric.

Exercici 6. Impost sobre el consum. Considera l'economia on totes les generacions $t \geq 1$ són idèntiques i on cada generació està formada per dos grups: el grup 1 i el grup 2. El grup 1 consta d'$N_1 = 300$ membres i cada membre jove de la generació t disposa de la dotació $(1,0)$ i té $u_1 = c_1(t) \cdot [c_1(t+1)]^2$ com a funció d'utilitat. El grup 2 està constituït per $N_2 = 100$ membres i en ell cada membre jove i de la generació t disposa de la dotació $(0,2)$ i té $u_2 = [c_2(t)]^2 \cdot c_2(t+1)$ com a funció d'utilitat. Cada període hi ha un impost de τ unitats del bé per unitat de bé consumida que ha de pagar cada jove del grup 1 i cada gran del grup 2. La recaptació de l'impost es distribueix igualitàriament entre el conjunt d'individus format pels joves del grup 2 i el grans del grup 1.

(i) Calcula l'equilibri general competitiu de l'economia i compara'l amb el que resultaria si no existís l'impost (ni la transferència). (ii) Determina el valor de τ que maximitza la suma de tots els membres de l'economia que viuen en un període determinat, assumint que la funció d'utilitat de cada individu gran coincideix amb el seu consum de gran.

Exercici 7. Pensions. Cada generació está formada per 40 individus amb dotació $(0,2)$ i 60 amb dotació $(1,0)$. La funció d'utilitat de cada consumidor i és $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$.

(i) Calcula l'equilibri general si només hi ha un mercat de préstecs.

(ii) El govern estableix un sistema de pensions de repartiment finançat amb un impost de 0,6 a pagar per cada jove amb dotació positiva quan és jove. La recaptació de l'impost a cada període es distribueix igualitàriament entre tots els individus grans del període. Calcula l'equilibri general.

(iii) El govern estableix un sistema de pensions de repartiment finançat amb un impost de τ unitats a pagar per cada jove amb dotació positiva quan és jove. La recaptació de l'impost a cada període es distribueix igualitàriament entre tots els individus grans del període. Calcula el valor de τ que fa que, en l'equilibri general, el consum de cada individu sigui el mateix valor c, tant si és jove com si és gran.

9. Bibliografia

• McCandless, G. T. i N. Wallace (1991): *Introduction to Dynamic Macroeconomic Theory: An Overlapping Generations Approach*, Harvard University Press, Harvard (MA).

Aquests apunts segueixen el capítol 3, on s'analitza l'establiment d'imposts sobre les dotacions, la creació d'un mercat de deute públic, la proposició d'equivalència ricardiana i l'equivalència entre equilibris amb deute i equilibris amb imposts.

• Ihori, Toshihiro (1996): *Public finance in an overlapping generations economy*, Macmillan Press, Londres.

Tot i que el nivell del text és més avançat, es poden consultar parts dels capítols 3 (imposts), 5 (despesa pública) i 10 (seguretat social) per a estudiar extensions i modificacions dels models més simples presentats en el capítol 3 de McCandless i Wallace.

• Wickens, Michael (2008): *Macroeconomic Theory: A Dynamic General Equilibrium Approach*, Princeton University Press, Princeton i Oxford.

• Heijdra, Ben J. (2009): *Foundations of Modern Macroeconomics*, 2a edició, Oxford University Press, Nova York.

En les pàgines 138–143 del manual de Wickens i les pàgines 624–647 del manual de Heijdra es desenvolupa l'anàlisi de les pensions.