Overlapping generations model for a Malthusian economy

1. Economic regimes

Galor and Weil (1999, p. 150; 2000, p. 806) characterize the process of economic development in terms of three regimes. In historical order, they are called Malthusian, Post-Malthusian, and Modern Growth Regimes.

According to their characterization, in the Malthusian Regime technological progress and population growth was almost negligible ("glacial by modern standards"), whereas income per capita (living standards) was nearly constant. In addition, there exists a positive relationship between income per capita and population growth: an increase in per capita income leads to an increase in population growth.

In the Post-Malthusian Regime income per capita grows but the positive relationship between per capita income and population growth still holds: a rising income per capita continues to lead to a rising population growth rate.

Lastly, the Modern Growth Regime is the opposite of the Malthusian Regime: technological level and income per capita steadily grow, at a higher rate than in the Post-Malthusian Regime, and the relationship between the level of income per capita and the population growth rate turns out to be negative, in the sense that a rising income per capita now leads to a declining population growth rate.

2. A Malthusian model

The purpose of this lesson is to present the model described in Ashraf and Galor (2011, pp. 2005-09) that replicates the basic features of the Malthusian Regime.

Only one good Y is produced in the economy, which operates in discrete time. There are two inputs: land D and labour L. The amount of land is fixed and constant in every period. The production function of the economy in period t is

$$Y(t) = (A \cdot D)^{\alpha} \cdot L(t)^{1-\alpha}$$

where:

- $Y(t)$ is the amount of the good produced in period t;
- A represents the state of the technology (the technology "level");
- D is the fixed amount of land;
- α is a number between 0 and 1; and
- $L(t)$ is the amount of labour employed in production in period t.

The term $A \cdot D$ can be viewed as expressing the resources that are effectively used in production. Loosely speaking, the state of the technology captures all the factors that make land more or less productive: soil quality, climate, cultivation methods, knowledge of techniques to apply land to production, etc. The combination $A \cdot D$ represents the effective land resources used in production: if $A = 2$, then having one unit of land is, in effect, like having two.

The agents providing labour are called farmers. The output per farmer $y(t)$ in period t is defined as

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{A \cdot D}{L(t)}\right)^{\alpha}.$$

Individuals live for two consecutive periods. All individuals are identical. In the second period of life (parenthood) every individual:

- decides how many children to have (families are single-parent);
- becomes a farmer and supplies labour inelastically (the farmer's supply of labour does not depend on income);
- the farmer's income in period t coincides with the output per farmer $y(t)$ in period t;
- a farmer's income is spent in consumption and in raising children.

In the first period of life (childhood), every individual is a child and, therefore, must be supported by his/her parent. As a child, an individual makes no decision.

The utility function $u(t)$ of an individual in his/her second period of life t is given by

$$u(t) = c(t)^{\beta} \cdot n(t)^{1-\beta}$$

where $c(t)$ is the amount of the good consumed by the individual in period t, $n(t)$ is the number of children that the individual has chosen to have, and β is a number between 0 and 1.

Raising a child has a fixed cost of $\gamma > 0$ units of the good per child. Each farmer from period t chooses $c(t)$ and $n(t)$ in order to maximize $u(t)$ facing the following budget constraint:

$$c(t) + \gamma \cdot n(t) = y(t).$$

To solve the problem

$$max_{c(t),n(t)}\ u(t) = c(t)^{\beta} \cdot n(t)^{1-\beta}$$

$$\text{subject to } c(t) + \gamma \cdot n(t) = y(t)$$

using the Lagrange multiplier technique, define

$$L(t) = c(t)^{\beta} \cdot n(t)^{1-\beta} + \lambda \cdot \big(y(t) - c(t) - \gamma \cdot n(t)\big)\,.$$

Then:

$$\frac{\partial L(t)}{\partial c(t)} = \beta \cdot c(t)^{\beta-1} \cdot n(t)^{1-\beta} - \lambda = 0$$

$$\frac{\partial L(t)}{\partial n(t)} = (1-\beta) \cdot n(t)^{-\beta} \cdot c(t)^{\beta} - \lambda \cdot \gamma = 0\,.$$

By the first equation,

$$\lambda = \beta \cdot c(t)^{\beta-1} \cdot n(t)^{1-\beta}.$$

By the second,

$$\lambda = \frac{1-\beta}{\gamma} \cdot c(t)^{\beta} \cdot n(t)^{-\beta}.$$

Therefore,

$$\beta \cdot c(t)^{\beta} \cdot \frac{1}{c(t)} \cdot n(t)^{-\beta} \cdot n(t) = \frac{1-\beta}{\gamma} \cdot c(t)^{\beta} \cdot n(t)^{-\beta}$$

and, consequently,

$$n(t) = \frac{1-\beta}{\beta \cdot \gamma} \cdot c(t)\,.$$

Inserting this into the budget constraint $c(t) + \gamma \cdot n(t) = y(t)$ yields

$$\boxed{c(t) = \beta \cdot y(t)\,.}$$

As a result,

$$\boxed{n(t) = \frac{1-\beta}{\gamma} \cdot y(t)\,.}$$

Since $1-\beta > 0$, an increase in output per farmer leads to an increase in the number of children: $\frac{dn(t)}{dy(t)} = \frac{1-\beta}{\gamma} > 0$. The model then reproduces the positive relationship between output per capita and population growth that is present in the Malthusian Regime.

3. Population dynamics and stationary state

For period t, let $L(t)$ designate the number of farmers in t (the working population). Consequently,

$$L(t+1) = n(t) \cdot L(t)\,.$$

This says that the number of famers in $t+1$ corresponds to the number of children that the farmers in t decided to have: the children of period t farmers become period $t+1$ farmers.

By combining the equations $L(t+1) = n(t) \cdot L(t)$, $n(t) = \frac{1-\beta}{\gamma} \cdot y(t)$, and $y(t) = \left(\frac{A \cdot D}{L(t)}\right)^{\alpha}$, it follows that

$$L(t+1) = \frac{1-\beta}{\gamma} \cdot \left(\frac{A \cdot D}{L(t)}\right)^{\alpha} \cdot L(t)$$

that is,

$$L(t+1) = \frac{1-\beta}{\gamma} \cdot (A \cdot D)^{\alpha} \cdot L(t)^{1-\alpha}$$

or, letting $a = \frac{1-\beta}{\gamma} \cdot (A \cdot D)^{\alpha}$,

$$\boxed{L(t+1) = a \cdot L(t)^{1-\alpha}\,.}$$

Since $a > 0$ and $0 < 1-\alpha < 1$,

$$\frac{dL(t+1)}{dL(t)} = a \cdot (1-\alpha) \cdot L(t)^{-\alpha} = \frac{a \cdot (1-\alpha)}{L(t)^{\alpha}} > 0$$

and

$$\frac{d^2L(t+1)}{dL(t)^2} = -a \cdot (1-\alpha) \cdot \alpha \cdot L(t)^{-\alpha-1} < 0\,.$$

The above derivatives mean that the function represented by $L(t+1) = a \cdot L(t)^{1-\alpha}$ is:

(i) increasing;

(ii) strictly concave;

(iii) intersects the origin ($L(t) = 0$ implies $L(t+1) = 0$); and

(iv) the first derivative $\frac{dL(t+1)}{dL(t)}$

(a) is decreasing;

(b) approaches zero as $L(t)$ goes to infinity; and

(c) goes to infinity as $L(t)$ converges to zero.

Fig. 1 below pictures $L(t+1) = a \cdot L(t)^{1-\alpha}$. Observe that, apart from the origin, the graph intersects the main diagonal (that is, the set of pairs $\big(L(t+1), L(t)\big)$ such that $L(t+1) = L(t)$) only once: at point e. At that point, corresponding to the value L^*, a stationary (or steady) state is reached, in the sense that $L(t) = L^*$ implies $L(t+1) = L^*$.

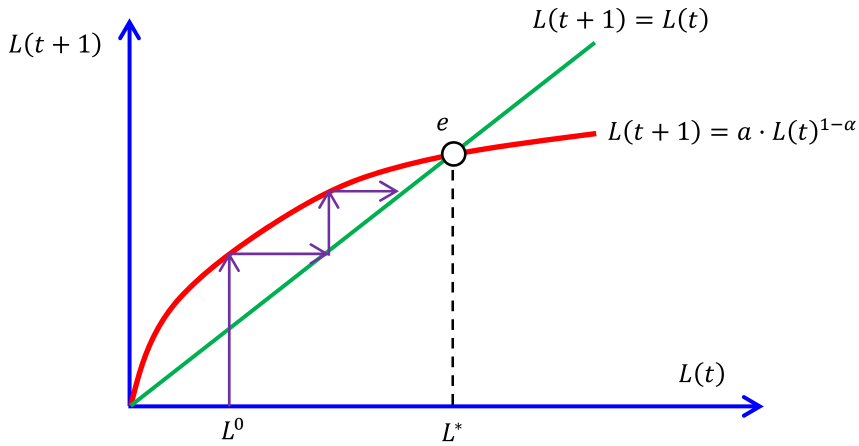

Fig. 1. Population dynamics and population steady state

The only stationary state (different from zero) associated with $L(t+1) = a \cdot L(t)^{1-\alpha}$ is obtained by setting $L(t+1) = L(t)$ and satisfies

$$L^* = a \cdot {L^*}^{1-\alpha}.$$

Solving for L^* yields the stationary value of population: $L^* = a^{1/\alpha}$. That is,

$$L^* = A \cdot D \cdot \left(\frac{1-\beta}{\gamma}\right)^{\frac{1}{\alpha}}. \tag{1}$$

The stationary state is stable in the same that, if the initial value of population is different from the stationary value L^*, the dynamics of the process $L(t+1) = a \cdot L(t)^{1-\alpha}$ makes population converge to L^*. This is illustrated in Fig. 1 by assuming that the initial value is $L^0 < L^*$: the sequence of values generated by the process approaches L^*. The same conclusion holds if $L^0 > L^*$.

Define the population density (or, properly, the farmers' density) in period t as $F(t) = L(t)/D$. The stationary density is then given by $F^* = L^*/D$ or

$$F^* = A \cdot \left(\frac{1-\beta}{\gamma}\right)^{\frac{1}{\alpha}}. \tag{2}$$

4. Output per capita dynamics and stationary state

Recalling that $y(t) = \left(\frac{A \cdot D}{L(t)}\right)^{\alpha}$ and $L(t+1) = n(t) \cdot L(t)$,

$$y(t+1) = \left(\frac{A \cdot D}{L(t+1)}\right)^{\alpha} = \left(\frac{A \cdot D}{n(t) \cdot L(t)}\right)^{\alpha} = \frac{1}{n(t)^{\alpha}} \cdot \left(\frac{A \cdot D}{L(t)}\right)^{\alpha} = \frac{y(t)}{n(t)^{\alpha}}.$$

Since $n(t) = \frac{1-\beta}{\gamma} \cdot y(t)$, it turns out that the following first-order difference equation describes the dynamics of output per capita:

$$y(t+1) = \left(\frac{\gamma}{1-\beta}\right)^{\alpha} \cdot y(t)^{1-\alpha}. \tag{3}$$

Setting $b = \left(\frac{\gamma}{1-\beta}\right)^{\alpha} > 0$, this can be expressed in a more compact form:

$$y(t+1) = b \cdot y(t)^{1-\alpha}.$$

This expression is analogous to the one describing the dynamics of the working population: $L(t+1) = a \cdot L(t)^{1-\alpha}$.

5. Exercises

Ejercicio 1. Estabilidad de un estado estacionario. Considera la Fig. 1. Explica si el estado estacionario representado por el origen (no hay población) es estable; es decir, asegura la dinámica de la población el retorno al origen si se produce una pertorbación que aleja mínimamente a la economía del origen?

Ejercicio 2. Estática comparativa. Partiendo de la Fig. 1: (a) analiza gráficamente el impacte sobre el estado estacionario (y, en particular, sobre L^* y F^*) de cada uno de los siguientes sucesos; (b) confirma el resultado del análisis gráfico estableciendo el signo de la derivadas parciales correspondientes de (1) y (2); (c) indica qué sucede con la renta per cápita de estado estacionario.

(i) Se produce una mejora tecnológica representada por un aumento del valor del parámetro A.

(ii) Se produce una regresión tecnológica representada por una disminución del valor del parámetro A.

(iii) Se incrementa el coste de tener hijos.

(iv) Aumenta el estoc de tierra D (se descubren territorios).

(v) Crece la preferencia por el consumo (el valor de β se incrementa).

(vi) Se reduce el valor del parámetro α.

Ejercicio 3. Teorema de Euler. Considera la función de producción estática $Y = (A \cdot D)^{\alpha} \cdot L^{1-\alpha}$.

(i) Calcula la función $\frac{\partial Y}{\partial D}$ de productividad marginal de la tierra.

(ii) Calcula la función $\frac{\partial Y}{\partial L}$ de productividad marginal de los agricultores.

(iii) Verifica que $Y = \frac{\partial Y}{\partial D} \cdot D + \frac{\partial Y}{\partial L} \cdot L$.

(iv) Sugiere una interpretación económica de la ecuación $Y = \frac{\partial Y}{\partial D} \cdot D + \frac{\partial Y}{\partial L} \cdot L$.

(v) Determina $\frac{\frac{\partial Y}{\partial D} \cdot D}{Y}$ e interpreta el resultado.

(vi) Asumiendo $A = 1$, obtén la derivada de la función $\frac{\partial Y}{\partial D}$ respecte de α e interpreta económicamente el signo.

Ejercicio 4. Dinámica de la renta per cápita. Determina la expresión que define el estado estacionario (no trivial) de la dinámica de la renta per cápita representada por (3) e identifica el signo de la derivada parcial respecto de cada parámetro.

Ejercicio 5. Predicciones malthusianas. Establece si las siguientes predicciones malthusianas se producen en el modelo explicado en el texto. [Ashraf y Galor (2011, p. 2009) observan que "These predictions emerge from a Malthusian model as long as the model is based upon two fundamental features: (i) a positive effect of the standard of living on population growth, and (ii) decreasing returns to labor due to the presence of a fixed factor of production—land."]

(i) Una mejora tecnológica que se traduce en una aumento de la productividad de la tierra (aumento de A) hace que, en el largo plazo, se incremente la población sin que se modifique el nivel de la renta per cápita.

(ii) Si la única diferencia estructural entre dos economías E y E' es que la productividad de la tierra (o el nivel tecnológico) de E es superior a la de E', entonces, a largo plazo, la economía E tendrá una densidad de población superior a la de E' pero no obtendrá de una renta per cápita mayor que E'.

6. References

Ashraf, Quamrul and Oded Galor (2011): "Dynamics and stagnation in the Malthusian epoch", American Economic Review 101(5), 2003-2041.

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