Overlapping generations with production, tax evasion, and transfers

The economy

1. Each generation 0 has 100 members: 50 of them ("the poor") with labour endowment $(1, 0)$ and the other 50 ("the rich") with labour endowment $(4, 0)$.

2. All (young) members of all generations have utility function $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$.

3. There is no capital. Production only depends on labour: $Y(t) = L(t)^{1/2}$.

4. To ensure that all output is distributed among workers (that is, $\omega(t) \cdot L(t) = Y(t)$), the wage rate is assumed to be twice the competitive wage rate: $\omega(t) = L(t)^{-1/2}$.

5. There is a government that sets a tax τ to be paid by the rich when young. For each t, the revenues of the tax at t are distributed among all the old at t (pay-as-you-go pension system). Each of the old will receive $\tilde{\tau}$.

6. The rich can spend a part x of their labour endowment to evade taxes. When a rich invest x in defrauding the government, he pays $\tau \cdot g(x)$ instead of τ, for some function g that takes non-neagive values and is increasing in x. Specifically, let $g(x) = \left(1 - \frac{x}{4}\right)^2$.

7. At each t, the government budget is balanced: the revenue collected from the rich equals the transfers made to the old $(100 \cdot \tilde{\tau})$. Revenue from the rich is not necessarily $50 \cdot \tau$ because it is not know the amount of tax evasion by the rich.

Finding $\tilde{\tau}$

Budget constraint of a young poor individual

$$c_t^{i,P}(t) + l^{i,P}(t) = \omega(t)$$

Budget constraint of an old, initially poor, individual

$$c_t^{i,P}(t+1) = R(t) \cdot l^{i,P}(t) + \tilde{\tau}$$

Lifetime budget constraint of a poor individual

$$c_t^{i,P}(t) + \frac{c_t^{i,P}(t+1)}{R(t)} = \omega(t) + \frac{\tilde{\tau}}{R(t)}$$

Budget constraint of a young rich individual

$$c_t^{i,R}(t) + \ l^{i,R}(t) \ + \tau \cdot \left(1 - \frac{x(t)}{4}\right)^2 = \omega(t) \cdot \big(4 - x(t)\big)$$

Budget constraint of an old, initially rich, individual

$$c_t^{i,R}(t+1) = \ R(t) \cdot l^{i,R}(t) + \tilde{\tau}$$

Lifetime budget constraint of a rich individual

$$c_t^{i,R}(t) + \frac{c_t^{i,R}(t+1)}{R(t)} = \ \omega(t) \cdot \big(4 - x(t)\big) - \ \tau \cdot \left(1 - \frac{x(t)}{4}\right)^2 + \frac{\tilde{\tau}}{R(t)}$$

Consumption function of a young poor individual

To maximize $u_t^{i,P}$, $MRS_t^{i,P} = R(t)$. Hence, $c_t^{i,P}(t) = \frac{c_t^{i,P}(t+1)}{R(t)}$. Using the lifetime budget constraint, $2 \cdot c_t^{i,P}(t) = \ \omega(t) + \frac{\tilde{\tau}}{R(t)}$. The demand function for consumption of a young poor individual is therefore $c_t^{i,P}(t) = \frac{\omega(t)}{2} + \frac{\tilde{\tau}}{2 \cdot R(t)}$.

Savings of a poor individual

The savings function of a young poor individual is $s^{i,P}(t) = \ \omega(t) - c_t^{i,P}(t) = \frac{\omega(t)}{2} - \frac{\tilde{\tau}}{2 \cdot R(t)}$.

Total savings of the poor individuals

Since there are 50 young poor individuals in period t, total savings $S^P(t)$ of the poor are

$$S^P(t) = 50 \cdot s^{i,P}(t) = 25 \cdot \omega(t) - \ 25 \cdot \frac{\tilde{\tau}}{R(t)}.$$

How much a young rich individual invests in evading the tax

The aim of every rich is maximize $u_t^{i,R}$ subject to his lifetime budget constraint. What is different with respect a young poor individual is that a rich individual chooses three values: $c_t^{i,R}(t)$, $c_t^{i,R}(t+1)$, and $x(t)$. The Lagrangian is

$$\mathcal{L} = c_t^{i,R}(t) \cdot c_t^{i,R}(t+1) + \lambda \left(\omega(t)\big(4 - x(t)\big) - \ \tau \cdot \left(1 - \frac{x(t)}{4}\right)^2 + \frac{\tilde{\tau}}{R(t)} - c_t^{i,R}(t) - \frac{c_t^{i,R}(t+1)}{R(t)}\right).$$

Deriving $\mathcal{L}$ with respect to $c_t^{i,R}(t)$ and $c_t^{i,R}(t+1)$ and equating to zero leads to the familiar condition $MRS_t^{i,R} = R(t)$. Accordingly, $c_t^{i,R}(t) = \frac{c_t^{i,R}(t+1)}{R(t)}$. On the other hand,

$$0 = \frac{\partial \mathcal{L}}{\partial x(t)} = \lambda \left(-\omega(t) - \ \tau \cdot 2 \cdot \left(-\frac{1}{4}\right) \cdot \left(1 - \frac{x(t)}{4}\right)\right).$$

That is,

$$x(t) = \ 4 - \frac{8 \cdot \omega(t)}{\tau}. \tag{1}$$

Balanced government budget condition

At t, the goverment pays $\tilde{\tau}$ to each of the 100 old individuals: total transfers equal $100 \cdot \tilde{\tau}$. Total revenue at t comes from the 50 rich individuals at t. Although the tax for each such individual is τ, what each rich individual actually pays, which is $\omega(t) \cdot \left(4 - x(t)\right) - \tau \cdot \left(1 - \frac{x(t)}{4}\right)^2$, depends on x. To balance transfers and revenues, $100 \cdot \tilde{\tau} = 50 \cdot \left(\omega(t) \cdot \left(4 - x(t)\right) - \tau \cdot \left(1 - \frac{x(t)}{4}\right)^2\right)$. Using (1),

$$2 \cdot \tilde{\tau} = \omega(t) \cdot \left(\frac{8 \cdot \omega(t)}{\tau}\right) - \tau \cdot \left(1 - \frac{4 - \frac{8 \cdot \omega(t)}{\tau}}{4}\right)^2 .$$

All in all,

$$\tilde{\tau} = \frac{2 \cdot \omega(t)^2}{\tau} . \qquad (2)$$

Interpreting equation (2)

Without tax evasion, an increase in τ entails an increase in $\tilde{\tau}$: the government can transfer more to the old by levying higher taxes on the rich. But, with tax evasion, there is an inverse relationship between τ and $\tilde{\tau}$: by (2), given $\omega(t)$, the higher the tax τ, the smaller the transfer $\tilde{\tau}$. This could be taken as an illustration of Goodhart's law: the attempt to control reality, changes reality itself. In this case, the rich react to the tax by trying to evade it, so that the government cannot expect to raise the planned revenue of $50 \cdot \tau$. (2) is also reminiscent of the Laffer curve: sufficiently high taxes are detrimental to tax collection.

Total savings of the poor individuals knowing (2)

By (2), total savings are actually $S^P(t) = 25 \cdot \omega(t) - 25 \cdot \frac{\tilde{\tau}}{R(t)} = 25 \cdot \omega(t) - 50 \cdot \frac{\omega(t)^2}{\tau \cdot R(t)}$.

Redefining the lifetime budget constraint of a rich individual

Inserting (1) and (2) in the lifetime budger constraint of the rich individual yields

$$c_t^{i,R}(t) + \frac{c_t^{i,R}(t+1)}{R(t)} = \omega(t) \cdot \left(\frac{8 \cdot \omega(t)}{\tau}\right) - \tau \cdot \left(1 - \frac{4 - \frac{8 \cdot \omega(t)}{\tau}}{4}\right)^2 + \frac{\frac{2 \cdot \omega(t)^2}{\tau}}{R(t)} ,$$

so

$$c_t^{i,R}(t) + \frac{c_t^{i,R}(t+1)}{R(t)} = \frac{4 \cdot \omega(t)^2}{\tau} + \frac{2 \cdot \omega(t)^2}{\tau \cdot R(t)} .$$

Consumption function of a young poor individual

Since $c_t^{i,R}(t) = \frac{c_t^{i,R}(t+1)}{R(t)}$, the demand function for consumption of a young rich individual is

$$c_t^{i,R}(t) = \frac{2 \cdot \omega(t)^2}{\tau} + \frac{\omega(t)^2}{\tau \cdot R(t)}.$$

Savings of a rich individual

By (1), the net income $\omega(t) \cdot \left(4 - x(t)\right) - \tau \cdot \left(1 - \frac{x(t)}{4}\right)^2$ of a rich individual is $\frac{4 \cdot \omega(t)^2}{\tau}$. Therefore, the savings of a young rich individual is

$$s^{i,R}(t) = \frac{4 \cdot \omega(t)^2}{\tau} - c_t^{i,R}(t) = \frac{2 \cdot \omega(t)^2}{\tau} - \frac{\omega(t)^2}{\tau \cdot R(t)}.$$

Total savings of the rich individuals

Since there are 50 young rich individuals in period t, total savings are

$$S^R(t) = 50 \cdot s^{i,R}(t) = 100 \cdot \frac{\omega(t)^2}{\tau} - 50 \cdot \frac{\omega(t)^2}{\tau \cdot R(t)}.$$

Total savings

The total savings function is $S(t) = S^P(t) + S^R(t)$. Hence,

$$S(t) = \left(25 \cdot \omega(t) - 50 \cdot \frac{\omega(t)^2}{\tau \cdot R(t)}\right) + \left(100 \cdot \frac{\omega(t)^2}{\tau} - 50 \cdot \frac{\omega(t)^2}{\tau \cdot R(t)}\right) =$$
$$= 25 \cdot \omega(t) + 100 \cdot \frac{\omega(t)^2}{\tau} - 100 \cdot \frac{\omega(t)^2}{\tau \cdot R(t)}.$$

Equilibrium condition

$$S(t) = 0$$

Equilibrium interest rate

Solving $25 \cdot \omega(t) + 100 \cdot \frac{\omega(t)^2}{\tau} - 100 \cdot \frac{\omega(t)^2}{\tau \cdot R(t)}$ for $R(t)$ yields

$$R(t) = \frac{4 \cdot \omega(t)}{4 \cdot \omega(t) + \tau}.$$

Wage rate

By assumption, $\omega(t) = L(t)^{-1/2}$. By (1), $L(t) = 50 \cdot 1 + 50 \cdot (4 - x(t)) = 50 + \frac{400 \cdot \omega(t)}{\tau}$. Therefore, $\omega(t) = L(t)^{-1/2} = \left(50 + \frac{400 \cdot \omega(t)}{\tau}\right)^{-1/2}$. Equivalently, $50 \cdot \omega(t)^2 + \frac{400 \cdot \omega(t)^3}{\tau} = 1$. This condition defines implicitly the relationship between the tax τ and the wage rate ω.

Case $\tau = 1$

The only non-imaginary solution has $\omega(t) = 0.1043$; see http://www.1728.org/cubic.htm. In view of this, $x(t) = 3.1656$ (79.14% of 4), $L(t) = 91.72$ (36.6% of the potential 250), and $\tilde{\tau} = 0.0217$ (so transfers constitute a mere 2% of the theoretical tax $\tau = 1$).