Overlapping generations with production

The economy

• Generation 0 has 100 members: $L(0) = 100$
• $L(t+1) = N \cdot L(t)$. More specifically, $L(t) = N^t \cdot L(0) = 100 \cdot N^t$.
• All members of all generations are alike, with $L^i = (1, 0)$ (one unit of labour when young and zero units when old) and $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$.
• Since all individuals are alike, there is no market for loans.

• The production function is $Y(t) = A(t) \cdot K(t)^{1/2} \cdot L(t)^{1/2}$.
• The wage rate is $\omega(t) = \frac{dY}{dL} = \frac{1}{2} \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{1/2}$.
• The price of capital is $\sigma(t) = \frac{dY}{dK} = \frac{1}{2} \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{-1/2}$.
• $\sigma(t+1)$ can be interpreted as the interest rate $R(t)$.

Case 1: neither population growth nor technological progress

Therefore, $N = 1$ and, for all t, $A(t) = A > 0$. For simplicity, set $A = 1$.

$K^i(t+1)$ is what individual i saves at t (in the form of capital) when young.

Budget constraint of each young individual: $c_t^i(t) + K^i(t+1) = 1 \cdot \omega(t)$.
Budget constraint of each old individual: $c_t^i(t+1) = \sigma(t+1) \cdot K^i(t+1)$.
Lifetime budget constraint of each individual: $c_t^i(t) + \frac{c_t^i(t+1)}{\sigma(t+1)} = \omega(t)$.

$MRS_t^i = \frac{du_t^i/dc_t^i(t)}{du_t^i/dc_t^i(t+1)} = \frac{c_t^i(t+1)}{c_t^i(t)}$.

To maximize u_t^i, $MRS_t^i = \sigma(t+1)$. Hence, $c_t^i(t) = \frac{c_t^i(t+1)}{\sigma(t+1)}$. Using the lifetime budgetr constraint, $2c_t^i(t) = \omega(t)$. Young i's demand function for consumption is $c_t^i(t) = \frac{\omega(t)}{2}$.

The savings function of individual i is $s^i(t) = \omega(t) - c_t^i(t) = \frac{\omega(t)}{2}$.

Since there are 100 young individuals in period t, total savings are $S(t) = 50 \cdot \omega(t) = 50 \cdot \frac{1}{2} \cdot \left(\frac{K(t)}{100}\right)^{1/2} = \frac{5}{2} \cdot K(t)^{1/2}$.

Equilibrium condition: $S(t) = K(t+1)$.

Hence, $K(t+1) = \frac{5}{2} \cdot K(t)^{1/2}$. This function is increasing $(\frac{dK(t+1)}{dK(t)} = \frac{5}{4} \cdot \frac{1}{K(t)^{1/2}} > 0)$ and concave $(\frac{d^2K(t+1)}{dK(t)^2} = -\frac{5}{8} \cdot \frac{1}{K(t)^{3/2}} < 0)$.

Condition for a steady state: $K(t+1) = K(t)$.

Steady-state $\bar{K} > 0$ satisfies $\bar{K} = \frac{5}{2} \cdot \bar{K}^{1/2}$. That is, $\bar{K} = \frac{25}{4}$. Accordingly, $\bar{K}^i = \frac{1}{16}$, $\bar{\omega} = \frac{1}{8}$, $\bar{\sigma} = 2$, $\bar{c}^{young} = \frac{1}{16}$, and $\bar{c}^{old} = \frac{1}{8}$.

Case 2: population growth without technological progress

The only change is that, now, $S(t) = L(t) \cdot s^i(t) = 100 \cdot N^t \cdot \frac{\omega(t)}{2} = 50 \cdot N^t \cdot \omega(t) = 50 \cdot N^t \cdot \frac{1}{2} \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{1/2} = 25 \cdot N^t \cdot \left(\frac{K(t)}{100 \cdot N^t}\right)^{1/2} = \frac{5}{2} \cdot K(t)^{1/2} \cdot \frac{N^t}{N^{t/2}} = \frac{5}{2} \cdot K(t)^{1/2} \cdot N^{t/2}$.

In equilibrium, $S(t) = K(t+1)$. As a result, $K(t+1) = \frac{5}{2} \cdot K(t)^{1/2} \cdot N^{t/2}$.

The (gross) rate of growth of K is $G_K(t+1) = \frac{K(t+1)}{K(t)} = \frac{\frac{5}{2} \cdot K(t)^{1/2} \cdot N^{t/2}}{\frac{5}{2} \cdot K(t-1)^{1/2} \cdot N^{(t-1)/2}} = \frac{K(t)^{1/2}}{K(t-1)^{1/2}} \cdot \frac{N^{t/2}}{N^{(t-1)/2}} = \left(\frac{K(t)}{K(t-1)}\right)^{1/2} \cdot N^{\frac{t}{2}-\left(\frac{t-1}{2}\right)} = G_K(t)^{1/2} \cdot N^{1/2}$. In the limit, $G_K(t+1) = G_K(t) = G_K$. Consequently, $G_K = {G_K}^{1/2} \cdot N^{1/2}$. That is, ${G_K}^{1/2} = N^{1/2}$. In sum, $G_K = N$: <u>the capital stock eventually accumulates at the same rate as the population</u>.

The (gross) rate of growth of Y is $G_Y(t+1) = \frac{Y(t+1)}{Y(t)} = \frac{A(t+1) \cdot K(t+1)^{1/2} \cdot L(t+1)^{1/2}}{A(t) \cdot K(t)^{1/2} \cdot L(t)^{1/2}} = \left(\frac{K(t+1)}{K(t)}\right)^{1/2} \cdot \left(\frac{100 \cdot N^{t+1}}{100 \cdot N^t}\right)^{1/2} = G_K(t+1)^{1/2} \cdot N^{1/2}$. In the limit, $G_Y(t+1) = G_Y(t) = G_Y$ and $G_K(t+1) = G_K(t) = G_K$. Therefore, $G_Y = {G_K}^{1/2} \cdot N^{1/2}$. As shown previously, $G_K = N$. It then follows that $G_Y = N^{1/2} \cdot N^{1/2} = N$: <u>total output eventually also accumulates at the same rate as the population</u>.

The (gross) rate of growth of output per capita $\frac{Y}{L}$ is $G_{Y/L}(t+1) = \frac{\frac{Y(t+1)}{L(t+1)}}{\frac{Y(t)}{L(t)}} = \frac{A(t+1) \cdot \left(\frac{K(t+1)}{L(t+1)}\right)^{1/2}}{A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{1/2}} = \left(\frac{K(t+1)}{K(t)}\right)^{1/2} \cdot \left(\frac{L(t+1)}{L(t)}\right)^{-1/2} = G_K(t+1)^{1/2} \cdot \left(\frac{100 \cdot N^{t+1}}{100 \cdot N^t}\right)^{-1/2} = G_K(t+1)^{1/2} \cdot N^{-1/2}$. In the limit, $G_{Y/L}(t+1) = G_{Y/L}(t) = G_{Y/L}$ and $G_K(t+1) = G_K(t) = G_K$. Hence, $G_{Y/L} = {G_K}^{1/2} \cdot N^{-1/2} = N^{1/2} \cdot N^{-1/2} = 1$. Since both Y and L grow at the same rate, $\frac{Y}{L}$ does not grow in the limit: <u>output per capita eventually stops growing and remains constant</u>.

Case 3: technological progress without population growth

Suppose $A(t) = A^t$: technology grows at rate A. Population is constant: for all t, $L(t) = 100$. Accordingly, $Y(t) = A^t \cdot K(t)^{1/2} \cdot L(t)^{1/2} = 10 \cdot A^t \cdot K(t)^{1/2}$ ·

As in Case 1, $s^i(t) = \omega(t) - c_t^i(t) = \frac{\omega(t)}{2}$ and $S(t) = 50 \cdot \omega(t)$, where $\omega(t) = \frac{1}{2} \cdot A^t \cdot K(t)^{1/2} \cdot L(t)^{-1/2} = \frac{1}{20} \cdot A^t \cdot K(t)^{1/2}$.

In equilibrium, $S(t) = K(t+1)$. In view of this, $K(t+1) = \frac{5}{2} \cdot K(t)^{1/2} \cdot A^t$.

The (gross) rate of growth of K is $G_K(t+1) = \frac{K(t+1)}{K(t)} = \frac{\frac{5}{2} \cdot K(t)^{1/2} \cdot A^t}{\frac{5}{2} \cdot K(t-1)^{1/2} \cdot A^{t-1}} = \frac{K(t)^{1/2}}{K(t-1)^{1/2}} \cdot A = A \cdot G_K(t)^{1/2}$. In the limit, $G_K(t+1) = G_K(t) = G_K$. As a consequence, $G_K = A \cdot {G_K}^{1/2}$. That is, $G_K = A^2$: <u>the capital stock eventually accumulates at a higher rate than technology</u>.

The (gross) rate of growth of Y is $G_Y(t+1) = \frac{Y(t+1)}{Y(t)} = \frac{10 \cdot A^{t+1} \cdot K(t+1)^{1/2}}{10 \cdot A^t \cdot K(t)^{1/2}} = A \cdot \left(\frac{K(t+1)}{K(t)}\right)^{1/2} = A \cdot G_K(t+1)^{1/2}$. In the limit, $G_Y(t+1) = G_Y(t) = G_Y$ and $G_K(t+1) = G_K(t) = G_K$. Thus, $G_Y = A \cdot {G_K}^{1/2}$. Since $G_K = A^2$, $G_Y = A \cdot (A^2)^{1/2} = A^2$: <u>total output eventually also accumulates at a higher rate than technology</u>.

The (gross) rate of growth of output per capita $\frac{Y}{L}$ is $G_{Y/L}(t+1) = \frac{\frac{Y(t+1)}{L(t+1)}}{\frac{Y(t)}{L(t)}} = \frac{Y(t+1)}{Y(t)} = G_Y(t+1)$. As population does not grow, it is plain that Y and $\frac{Y}{L}$ both grow at the same rate. In particular, in the limit, <u>output per capita grows at a higher rate than technology</u>.

Case 4: population growth with technological progress

Let $Y(t) = A^t \cdot K(t)^{1/2} \cdot L(t)^{1/2}$ and $L(t) = 100 \cdot N^t$. Now:

$$\omega(t) = \frac{1}{2} \cdot A^t \cdot K(t)^{1/2} \cdot L(t)^{-1/2} = \frac{1}{20} \cdot A^t \cdot \frac{K(t)^{1/2}}{N^{t/2}}$$

$$S(t) = L(t) \cdot s^i(t) = 100 \cdot N^t \cdot \frac{\omega(t)}{2} = 100 \cdot N^t \cdot \frac{1}{20} \cdot A^t \cdot \frac{K(t)^{1/2}}{N^{t/2}} = \frac{5}{2} \cdot A^t \cdot K(t)^{1/2} \cdot N^{t/2}$$

In equilibrium, $S(t) = K(t+1)$. Accordingly:

$$K(t+1) = \frac{5}{2} \cdot A^t \cdot K(t)^{1/2} \cdot N^{t/2}$$

With $G_K(t+1) = \frac{K(t+1)}{K(t)} = \frac{\frac{5}{2} \cdot A^t \cdot K(t)^{1/2} \cdot N^{t/2}}{\frac{5}{2} \cdot A^{t-1} \cdot K(t-1)^{1/2} \cdot N^{(t-1)/2}} = A \cdot G_K(t)^{1/2} \cdot N^{\frac{t}{2} - \left(\frac{t-1}{2}\right)} = A \cdot G_K(t)^{1/2} \cdot N^{1/2}$, in the limit, $G_K = A \cdot {G_K}^{1/2} \cdot N^{1/2}$, so ${G_K}^{1/2} = A \cdot N^{1/2}$. All in all, $G_K = A^2 \cdot N$: the rate at which the capital stock eventually accumulates is the rate at which population grows multiplied by the square of the rate at which technology progresses.

With $G_Y(t+1) = \frac{Y(t+1)}{Y(t)} = \frac{A^{t+1} \cdot K(t+1)^{1/2} \cdot L(t+1)^{1/2}}{A^t \cdot K(t)^{1/2} \cdot L(t)^{1/2}} = A \cdot G_K(t+1)^{1/2} \cdot \left(\frac{100 \cdot N^{t+1}}{100 \cdot N^t}\right)^{1/2} = A \cdot G_K(t+1)^{1/2} \cdot N^{1/2}$, in the limit $G_Y = A \cdot {G_K}^{1/2} \cdot N^{1/2} = A \cdot (A^2 \cdot N)^{1/2} \cdot N^{1/2} = A^2 \cdot N$. This says that the rate at which total output eventually accumulates is the rate at which population grows multiplied by the square of the rate at which technology progresses.

With $G_{Y/L}(t+1) = \frac{\frac{Y(t+1)}{L(t+1)}}{\frac{Y(t)}{L(t)}} = \frac{A^{t+1} \cdot \left(\frac{K(t+1)}{L(t+1)}\right)^{1/2}}{A^t \cdot \left(\frac{K(t)}{L(t)}\right)^{1/2}} = A \cdot \left(\frac{\frac{K(t+1)}{100 \cdot N^{t+1}}}{\frac{K(t)}{100 \cdot N^t}}\right)^{1/2} = A \cdot \left(\frac{\frac{K(t+1)}{K(t)}}{\frac{100 \cdot N^{t+1}}{100 \cdot N^t}}\right)^{1/2} = A \cdot \left(\frac{G_K(t+1)}{N}\right)^{1/2}$, in the limit, $G_{Y/L} = A \cdot {G_K}^{1/2} \cdot N^{-1/2} = A \cdot (A^2 \cdot N)^{1/2} \cdot N^{-1/2} = A^2$: the rate at which output per capita eventually accumulates is the square of the rate at which technology accumulates.