The Solow-Swan growth model

- The economy evolves in discrete time. Time is measured in periods, denoted by t, and indexed by integers: $t \in \{1, 2, 3, \dots\}$.
- There is only one good in each period. The good is produced. The amount of good in t is $Y(t)$. Its price is normalized to 1 in every t.
- There are two types of agents: consumers (individuals or families) and firms.
- Consumers are globally characterized by a constant saving rate $\mathbb{s}$: in total, each t, consumers save a fixed proportion $\mathbb{s}$ of $Y(t)$.

- Consumers supply labour inelastically in a competitive market in exchange for a wage rate $\omega(t)$. Labour is identified with employment. The total supply of labour (total employment) in period t is $L(t)$ and is assumed to grow at a constant rate n.

- Firms are represented by a profit-maximizing aggregate firm that produces the good using an aggregate production function $Y(t) = F(A(t), K(t), L(t))$ exhibiting constant returns to scale [alternative interpretation: all firms have access to the same (free) production technology].

- $K(t)$ is the capital stock: good used to produce more good. The good can be consumed by consumers or used in production by firms.
- $A(t)$ represents the state of technology at time t. By convenience, it is defined by a number but it has no natural units. It is rather an index for a family of production functions that depend on just K and L.
- $A(t)$ captures everything affecting the way of using K and L efficiently.

Axioms on the production function F

- Takes non-negative values: $F \geq 0$.
- Is twice differentiable with respect to K & L.
- Positive marginal productivities of K & L: $F_K \coloneqq \frac{\partial F}{\partial K} > 0$ and $F_L \coloneqq \frac{\partial F}{\partial L} > 0$.
- Decreasing marginal productivities of K & L: $F_{KK} \coloneqq \frac{\partial^2 F}{\partial K^2} < 0$ and $F_L \coloneqq \frac{\partial^2 F}{\partial L^2} < 0$.
- Exhibits constant returns to scale (is homogeneous of degree 1) in K & L.
- F homogeneous of degree h in K & L if, for all $\lambda > 0$ and A, $F(A, \lambda K, \lambda L) = \lambda^h F(A, K, L)$.
- The above assumptions make F concave.

Euler's theorem

➡ *Euler's homogeneous function theorem (Euler's adding-up theorem). If $F(A, K, L)$ is twice differentiable with respect to K and L, and homogeneous of degree h in K and L, then:*

(i) for all (A, K, L), $h \cdot F(A, K, L) = F_K \cdot K + F_L \cdot L$;

(ii) F_K & F_L are homogeneous of degree $h - 1$ in K & L.

- Product exhaustion: if $h = 1$, $F(A, K, L) = F_K \cdot K + F_L \cdot L$ means that output F is exhausted if factors of production are paid according to their marginal productivities.

Additional assumptions

- All markets (for output Y, capital K, labour L) are competitive. The price of K and L at t are denoted, respectively, by $\omega(t)$ and $\sigma(t)$.
- The capital stock K depreciates at a constant rate $\delta > 0$: 1 unit of K at time t, becomes $1 - \delta$ units at time $t + 1$.
- The (real) interest rate is $r(t) = \sigma(t) - \delta$. If one unit of the good is used as capital at (the start of) t, then, at the end of t, its marginal productivity (equal to $\sigma(t)$, see 7) is obtained, but a fraction δ of the capital is lost.

Firms

- Firms maximize profits and can be represented by a unique firm with production function F. Hence, the firms' decisions at t are determined by the solution to the problem

$$maximize_{K,L}\ \ F(A(t),K,L)-\sigma(t)K-\omega(t)L$$

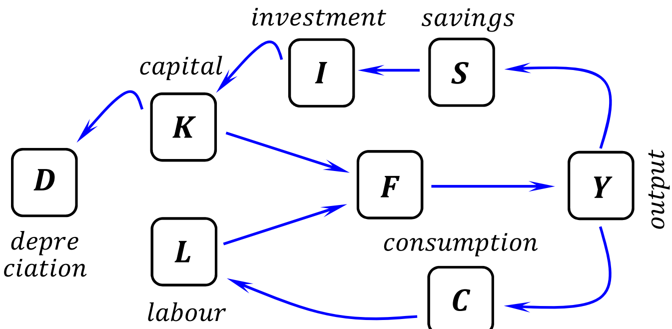
  where $A(t)$, $\sigma(t)$, and $\omega(t)$ are given and the price of output $Y(t)$ at t is normalized to 1.

- As F is concave, by the first order condition, $\sigma(t)=F_K$ and $\omega(t)=F_L$. By Euler's theorem, $Y(t)=\sigma(t)K(t)+\omega(t)L(t)$, so profits are 0.

(Ken-Ichi) Inada's conditions

- $\lim_{K\to 0} F_K = \infty$ and $\lim_{K\to\infty} F_K = 0$
- $\lim_{L\to 0} F_L = \infty$ and $\lim_{L\to\infty} F_L = 0$
- Initial units of an input are highly productive.
- For a sufficiently high amount of input, additional units are unproductive.
- Helpful to ensure interior solutions.
- No satisfied by production functions that are linear in K or L (like $F = A \cdot (K + L)$).

The economy's life cycle



- Macro <u>equilibrium condition</u>: $Y(t) = C(t) + I(t)$ or $S(t) = I(t)$, where $S(t) = \mathbb{s} \cdot Y(t)$
- Capital stock <u>accumulation condition</u>: $K(t+1) = I(t) + (1-\delta) \cdot K(t)$

Equilibrium path

- Model defined by the following equations.
  - Production function
    $Y(t) = F(A, K(t), L)$
  - Uses of output
    $Y(t) = C(t) + I(t)$
  - Savings function
    $S(t) = \mathbb{s} \cdot Y(t)$, $0 < \mathbb{s} < 1$
    (and, therefore, $C(t) = (1 - \mathbb{s}) \cdot Y(t)$)
  - Macroeconomic equilibrium
    $S(t) = I(t)$
  - Capital stock accumulation
    $\Delta K(t) = I(t) - \delta \cdot K(t) = \mathbb{s} \cdot Y(t) - \delta \cdot K(t)$

Equilibrium path

- Given population L, an initial capital stock $K(1)$, and technology represented by A, an equilibrium path for the economy is a sequence $\{K(t), Y(t), C(t), \sigma(t), \omega(t)\}_{t\geq 1}$ such that:

(i) $K(t+1) = \mathbb{s} \cdot F(A, K(t), L) + (1-\delta) \cdot K(t)$

(ii) $Y(t) = F(A, K(t), L)$

(iii) $C(t) = (1-\mathbb{s}) \cdot Y(t)$

(iv) $\sigma(t) = F_K$

(v) $\omega(t) = F_L$.

Per capita variables

- Define capital per capita (or capital-labour ratio) as $k(t) = \frac{K(t)}{L}$ and output per capita as $y(t) = \frac{Y(t)}{L}$.

- By constant returns,

$$y(t) = \frac{Y(t)}{L} = \frac{1}{L} F(A, K(t), L) = F\left(A, \frac{K(t)}{L}, \frac{L}{L}\right) =$$

$$= F(A, k(t), 1) = f(k(t))$$

- This says that output per capita is a function of capital per capita. As shown in 7,

$$\sigma(t) = \frac{\partial F}{\partial (K(t)/L)} = f'(k(t))\,.$$

- By Euler's theorem,

$$\omega(t) = f(k(t)) - k(t)f'(k(t))\,.$$

- The dynamics of capital accumulation is $K(t+1) = \mathbb{s} \cdot F(A, K(t), L) + (1-\delta) \cdot K(t)$. If both sides are divided by L,

$$k(t+1) = \mathbb{s} \cdot f(k(t)) + (1-\delta) \cdot k(t)$$

  which represents the dynamics of capital per capita accumulation.

- The above equation summarizes the model when there is neither population growth nor technological progress.

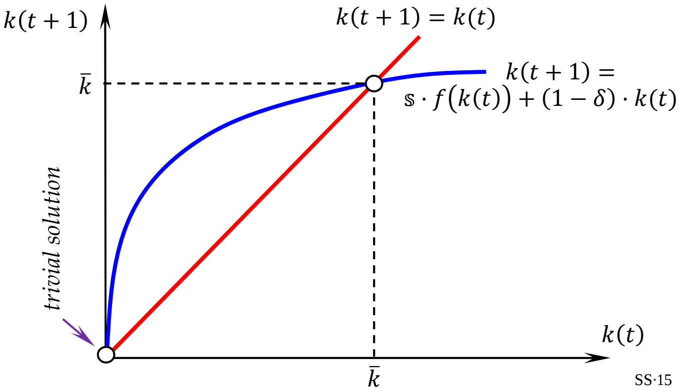
Solving the model

- A <u>steady-state equilibrium</u> (without population growth nor technological progress) is sn equilibrium path where, for some $\bar{k}$ and all t, $k(t) = \bar{k}$ (capital per capita remains constant).
- As L is constant, k constant implies that K constant (hence, Y and y are also constant).
- Steady-state equilibria are obtained from $k(t+1) = \mathbb{s} \cdot f(k(t)) + (1-\delta) \cdot k(t)$ by setting $k(t+1) = k(t) = \bar{k}$. Thus, $\bar{k} \neq 0$ satisfies

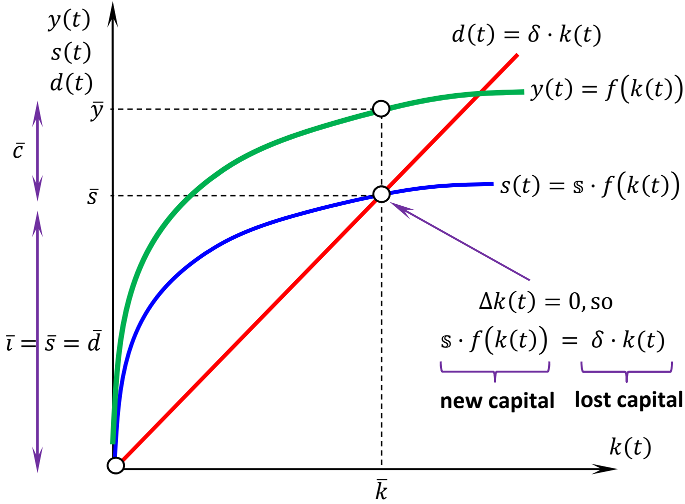
$$\frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{\mathbb{s}}.$$

Existence of steady-state equilibrium

- *Given the assumptions on* F*, there is a unique* $\bar{k} \neq 0$ *such that* $\frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{\mathbb{s}}$*, with output per capita* $\bar{y} = f\left(\bar{k}\right)$ *and consumption per capita* $\bar{c} = (1 - \mathbb{s}) \cdot f(\bar{k})$.

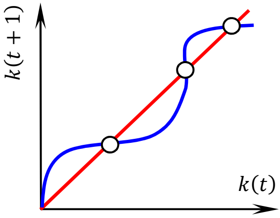


Another graphical representation

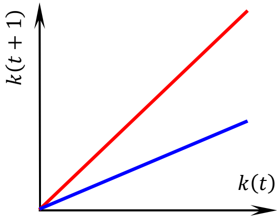


Non-existence & non-uniqueness

- In case (i), f fails to be concave (marginal productivities need not be decreasing). Several interior steady states may exist.
- Case (ii) shows that the failure of the Inada conditions may be lead to the non-existence of interior steady states.



(i)



(ii)

Stability of steady-state equilibrium

- *Let* $g: \mathbb{R} \to \mathbb{R}$ *be differentiable and* $g(\bar{x}) = \bar{x}$*. If, for all* x*,* $|g'(x)| < 1$*, then the steady state* $\bar{x}$ *of* $x(t+1) = g\big(x(t)\big)$ *is globally asymptotically stable; that is,* $\{x(t)\}$ *converges to* $\bar{x}$*.*

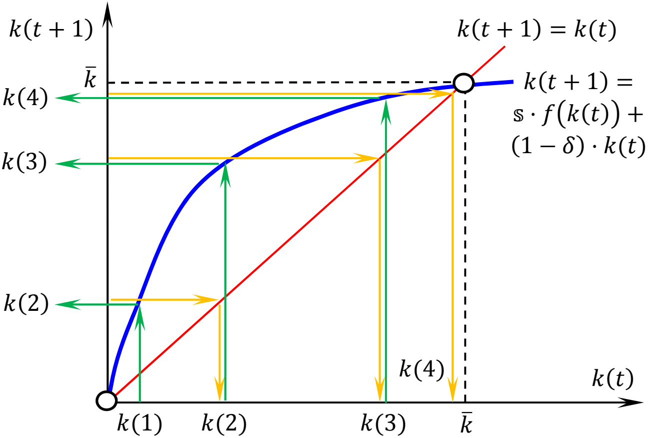
- *Given the assumptions on* F*:*

  (i) *the steady-state equilibrium of* $k(t+1) = \mathbb{s} \cdot f\big(k(t)\big) + (1-\delta) \cdot k(t)$*, which corresponds to* $k = \bar{k}$*, is* *__globally asymptotically stable__; and*

  (ii) *for all* $k(1) > 0$*,* $\{k(t)\}$ *converges* *__monotonically__* *to* $\bar{k}$.

Convergence to steady state ($k(1) < \bar{k}$)

- The dynamics of capital accumulation is similar to the dynamics in the OLG model.



Prices of inputs

- If $k(1) < \bar{k}$, then the sequence of wages $\{\omega(t)\}_{t\geq 1}$ is increasing, whereas $\{\sigma(t)\}_{t\geq 1}$ is a decreasing sequence ($\uparrow k \Rightarrow \downarrow f' \Rightarrow \downarrow \sigma$).
- If $k(1) < \bar{k}$, then convergence to $\bar{k}$ means that capital accumulates (k increases). Since $\uparrow k$ implies $\uparrow f$ and $\downarrow f'$, it follows from $\omega = f - k \cdot f'$ that ω falls. Intuitively, more k makes labour more productive, so ω rises.
- If $k(1) > \bar{k}$, capital decumulates, so $\{\omega(t)\}_{t\geq 1}$ is decreasing and $\{\sigma(t)\}_{t\geq 1}$ is increasing.

Comparative statics

- $\frac{f(k)}{k}$ is a <u>decreasing function of k</u>: $\frac{\partial\left(\frac{f(k)}{k}\right)}{\partial k} < 0$.
- Changes in the <u>depreciation rate</u>. $\uparrow \delta \Rightarrow \downarrow \bar{k}$: since $\frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{\mathbb{s}}, \uparrow \delta \Rightarrow \uparrow \frac{\delta}{\mathbb{s}} \Rightarrow \uparrow \frac{f(\bar{k})}{\bar{k}} \Rightarrow \downarrow \bar{k}$.
- Changes in the <u>savings rate</u>. $\uparrow \mathbb{s} \Rightarrow \uparrow \bar{k}$: as $\frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{\mathbb{s}}, \uparrow \mathbb{s} \Rightarrow \downarrow \frac{\delta}{\mathbb{s}} \Rightarrow \downarrow \frac{f(\bar{k})}{\bar{k}} \Rightarrow \uparrow \bar{k}$.
- A <u>shift upwards of f</u> to g yields $\uparrow \bar{k}$: given $\frac{g(\bar{k}_g)}{\bar{k}_g} = \frac{\delta}{\mathbb{s}} = \frac{f(\bar{k}_f)}{\bar{k}_f}$, $\frac{g(\bar{k}_f)}{\bar{k}_f} > \frac{\delta}{\mathbb{s}}$ follows from $g(\bar{k}_f) > f(\bar{k}_f)$, so $\uparrow \bar{k}$ is needed to get $\downarrow \frac{g(\bar{k}_f)}{\bar{k}_f}$. Therefore, $\bar{k}_g > \bar{k}_f$.

Capital accumulation growth rate

- Letting $\Delta k(t) \coloneqq k(t+1) - k(t)$, the equation

  $$k(t+1) = \mathbb{s} \cdot f(k(t)) + (1-\delta) \cdot k(t)$$

  can be equivalently expressed as

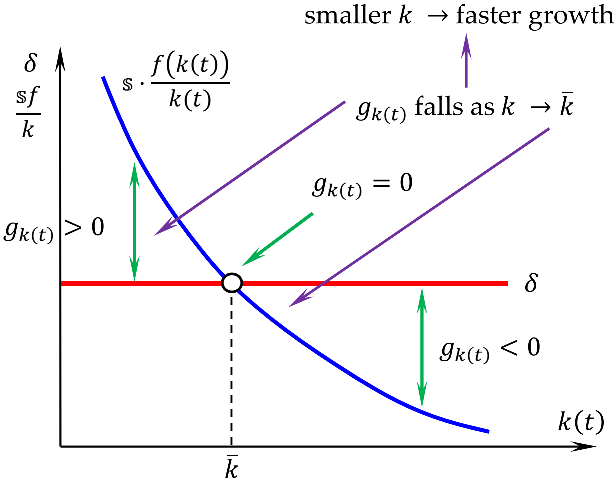
  $$\Delta k(t) = \mathbb{s} \cdot f(k(t)) - \delta \cdot k(t).$$

- Then the growth rate $g_{k(t)}$ of $k(t)$ is

  $$g_{k(t)} = \frac{\Delta k(t)}{k(t)} = \mathbb{s}\frac{f(k(t))}{k(t)} - \delta = \mathbb{s}\frac{y(t)}{k(t)} - \delta.$$

- Therefore, $g_{k(t)} > 0$ (k accumulates) if and only if $\mathbb{s} \cdot \frac{f(k)}{k} > \delta$. With $k(1) < \bar{k}$, convergence to $\bar{k}$ implies that $g_{k(t)}$ falls and goes to 0.

Representing $g_{k(t)}$ graphically



Golden rule of capital accumulation

- Steady-state per capita consumption $\bar{c}$, defined as a function of the savings rate $\mathbb{s}$, is

$$\bar{c}(\mathbb{s}) \coloneqq (1-\mathbb{s}) \cdot f\left(\bar{k}(\mathbb{s})\right) = f\left(\bar{k}(\mathbb{s})\right) - \mathbb{s} \cdot f\left(\bar{k}(\mathbb{s})\right) = f\left(\bar{k}(\mathbb{s})\right) - \delta \cdot \bar{k}(\mathbb{s}).$$

- The golden rule savings rate $\mathbb{s}^*$ maximizes steady-state per capita consumption $\bar{c}$.

$$0 = \frac{d\bar{c}(\mathbb{s})}{d\mathbb{s}} = f'\left(\bar{k}(\mathbb{s})\right)\frac{d\bar{k}(\mathbb{s})}{d\mathbb{s}} - \delta \cdot \frac{d\bar{k}(\mathbb{s})}{d\mathbb{s}} = \frac{d\bar{k}(\mathbb{s})}{d\mathbb{s}}\left(f'\left(\bar{k}(\mathbb{s})\right) - \delta\right).$$

- Therefore, $\mathbb{s}^*$satisfies $f'\left(\bar{k}(\mathbb{s}^*)\right) = \delta$.

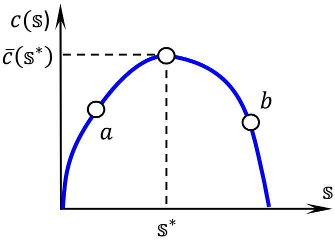
- $\bar{k}(\mathbb{s}^*)$ can be computed from $f' = \delta$. Given $\bar{k}(\mathbb{s}^*)$ and the steady-state condition

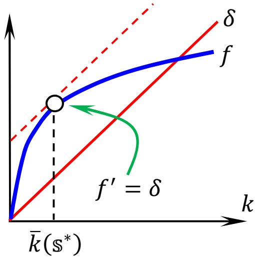
$$\mathbb{s}^* \cdot f\left(\bar{k}(\mathbb{s}^*)\right) = \delta \cdot \bar{k}(\mathbb{s}^*).$$

$\mathbb{s}^*$ satisfies

$$\mathbb{s}^* = \frac{\bar{k} \cdot f'(\bar{k})}{f(\bar{k})},$$

which is the share of k in y (in the steady-state). Graphically:





Golden rule as equilibrium in returns

- Consider capital K and labour L as two assets produced in the economy.
- The rate of return of capital can be defined as $f' - \delta$: the productivity of capital minus the depreciation of capital.
- The rate of return of labour can be taken to be zero: population does not grow.
- If both returns are to be the same, $f' - \delta = 0$; that is, $f' = \delta$.

Dynamic inefficiency

- An economy is dynamically inefficient if per capita consumption can be increased by reducing the capital stock (someone is better off, and no one worse off, with less capital).

➡ *If $\mathbb{s} > \mathbb{s}^*$, then (at the corresponding steady state) the economy is dynamically inefficient.*

- Capital overaccumulation (b in 25) generates dynamic inefficiency (a does not). If $\mathbb{s}_b$ is reduced to $\mathbb{s}^*$, per capita consumption is higher during the transition to the new steady state (immediate effect) and at the new steady state (permanent effect).

Dynamics of k with population growth

- Let $L(t+1) = (1+n) \cdot L(t)$. The condition for capital stock accumulation is, as before,

$$K(t+1) = \mathbb{s} \cdot Y(t) + (1-\delta) \cdot K(t)\,.$$

- Dividing both sides by $L(t+1)$,

$$k(t+1) = \mathbb{s} \cdot \frac{Y(t)}{L(t+1)} + (1-\delta) \cdot \frac{K(t)}{L(t+1)}\,.$$

- That is,

$$k(t+1) = \frac{\mathbb{s}}{1+n} \cdot y(t) + \frac{1-\delta}{1+n} \cdot k(t)$$

or

$$\Delta k(t) = \frac{\mathbb{s}}{1+n} \cdot f(k(t)) - \frac{\delta+n}{1+n} \cdot k(t)\,.$$

Steady state with population growth

- In a steady state, $\Delta k(t) = 0$. Therefore, capital per capita $\bar{k}$ in a steady state satisfies

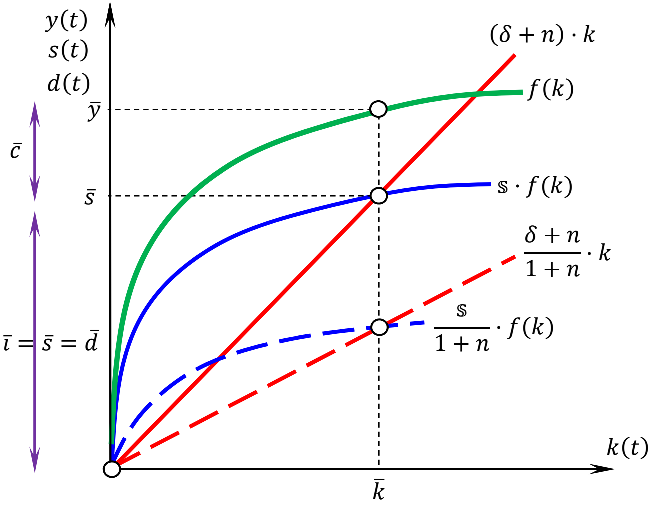
$$\frac{f(\bar{k})}{\bar{k}} = \frac{\delta + n}{\mathbb{s}}.$$

- Despite the presence of the denominator $1 + n$, $\bar{k}$ can be calculated by equating $\mathbb{s} \cdot y(t)$ with $(\delta + n) \cdot k(t)$; see 30.

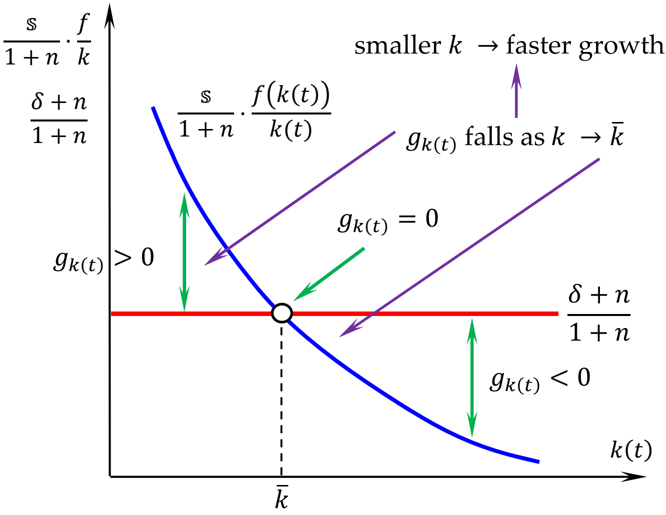
- The growth rate $g_{k(t)}$ of $k(t)$ is now (see 31)

$$g_{k(t)} = \frac{\mathbb{s}}{1+n} \frac{f\big(k(t)\big)}{k(t)} - \frac{\delta + n}{1+n}.$$

Depicting the steady state when $n > 0$



Growth rate $g_{k(t)}$ when $n > 0$



Steady state values

- In a steady state, no matter whether $n = 0$ or $n > 0$, per capita variables remain constant: y, k, $c\ldots$
- But, when $n = 0$, aggregate variables (Y, K, $C\ldots$) also remain constant, because L is constant: k and L constant imply K constant, y and L constant imply Y constant, etc.
- When $n > 0$, $k = K/L$ constant and L growing at rate n imply K growing at rate n. By constant returns, if K and L grow at rate n, then Y also grows at rate n.

The golden rule with $n > 0$

- In a steady state, consumption per capita is
$$\bar{c}(\mathbb{s}) \coloneqq (1-\mathbb{s}) \cdot f\left(\bar{k}(\mathbb{s})\right) = f\left(\bar{k}(\mathbb{s})\right) - \mathbb{s} \cdot f\left(\bar{k}(\mathbb{s})\right) = f\left(\bar{k}(\mathbb{s})\right) - (\delta + n) \cdot \bar{k}(\mathbb{s}).$$
- The <u>golden rule savings rate</u> $\mathbb{s}^*$ maximizes steady-state per capita consumption $\bar{c}$.

$$0 = \frac{d\bar{c}(\mathbb{s})}{d\mathbb{s}} = f'\left(\bar{k}(\mathbb{s})\right)\frac{d\bar{k}(\mathbb{s})}{d\mathbb{s}} - (\delta+n) \cdot \frac{d\bar{k}(\mathbb{s})}{d\mathbb{s}} = \\ = \frac{d\bar{k}(\mathbb{s})}{d\mathbb{s}}\left(f'^{\left(\bar{k}(\mathbb{s})\right)} - (\delta+n)\right)$$

- Therefore, $\mathbb{s}^*$satisfies $f'\left(\bar{k}(\mathbb{s}^*)\right) = \delta + n$.

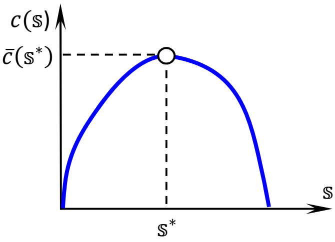
- $\bar{k}(\mathbb{s}^*)$ can be computed from $f' = \delta + n$. Using $\bar{k}(\mathbb{s}^*)$ and the steady-state condition

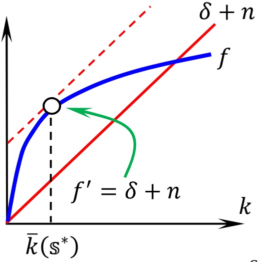
$$\mathbb{s}^* \cdot f\left(\bar{k}(\mathbb{s}^*)\right) = (\delta + n) \cdot \bar{k}(\mathbb{s}^*).$$

$\mathbb{s}^*$ satisfies

$$\mathbb{s}^* = \frac{\bar{k} \cdot f'(\bar{k})}{f(\bar{k})}.$$

- The situation is analogous to the one arising when $n = 0$.





Types of neutral technological progress

- Hicks-neutral technological progress means that $F(A, K, L)$ can be expressed, for some G, as $A \cdot G(K, L)$. A technological change just relabels isoquants (no change in shape).

- Solow-neutral (or capital-augmenting) technological progress means that $F(A, K, L)$ can be written as $G(A \cdot K, L)$, for some G. Higher A is equivalent to having more K.

- Harrod-neutral (or labor-augmenting) technological progress makes $F(A, K, L)$ equal to $G(K, A \cdot L)$, for some G (higher A is like more L).

Population and technology growth

- Population grows at rate n and technology accumulates at a constant rate a: $L(t+1) = (1+n) \cdot L(t)$ and $A(t+1) = (1+a) \cdot A(t)$.

- Technological progress is assumed to be Harrod-neutral. Define per capita variables in terms of efficiency units of labour $A \cdot L$:

$$k(t) = \frac{K(t)}{A(t) \cdot L(t)} \qquad y(t) = \frac{Y(t)}{A(t) \cdot L(t)}.$$

- Capital stock accumulates following the rule

$$K(t+1) = \mathbb{s} \cdot Y(t) + (1-\delta) \cdot K(t).$$

- Dividing both sides by $A(t+1) \cdot L(t+1)$,

$$k(t+1) = \frac{\mathbb{s} \cdot Y(t)}{A(t+1) \cdot L(t+1)} + \frac{(1-\delta) \cdot K(t)}{A(t+1) \cdot L(t+1)} =$$

$$= \frac{\mathbb{s}}{(1+a)(1+n)} \cdot \frac{Y(t)}{A(t) \cdot L(t)} + \frac{1-\delta}{(1+a)(1+n)} \cdot \frac{K(t)}{A(t) \cdot L(t)} =$$

$$= \frac{\mathbb{s}}{(1+a)(1+n)} \cdot y(t) + \frac{1-\delta}{(1+a)(1+n)} \cdot k(t)$$

- In sum,

$$k(t+1) = \frac{\mathbb{s}}{(1+a)(1+n)} \cdot y(t) + \frac{1-\delta}{(1+a)(1+n)} \cdot k(t)$$

or, equivalently,

$$\Delta k(t) = \frac{\mathbb{s}}{(1+a)(1+n)} \cdot f(k(t)) - \frac{\delta + n + a(1+n)}{(1+a)(1+n)} \cdot k(t) \,.$$

Steady state with $n > 0$ and $a > 0$

- In a steady state, $\Delta k(t) = 0$. Accordingly, capital per capita $\bar{k}$ in a steady state satisfies

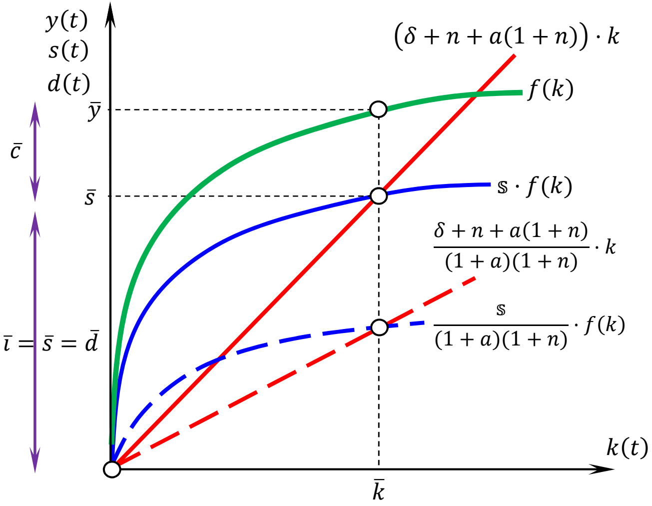
$$\frac{f(\bar{k})}{\bar{k}} = \frac{\delta + n + a(1+n)}{\mathbb{s}}.$$

- Despite the presence of $(1+a)(1+n)$ in the denominator, $\bar{k}$ can be obtained by equating $\mathbb{s} \cdot f(k(t))$ with $(\delta + n + a(1+n)) \cdot k(t)$.

- The growth rate $g_{k(t)}$ of $k(t)$ is now

$$g_{k(t)} = \frac{\mathbb{s}}{(1+a)(1+n)} \frac{f\big(k(t)\big)}{k(t)} - \frac{\delta + n + a(1+n)}{(1+a)(1+n)}.$$

Graph of the steady state, $n > 0$ & $a > 0$



Effect of technological progress

- Since k converges to $\bar{k}$, output per efficiency unit of labour y also converges (to some $\bar{y}$). This means that output per efficiency unit does not grow in the long run.

- Formally, $\frac{K}{A \cdot L}$ is eventually constant ($\bar{k}$). Denoting growth rates by g, $\frac{K}{A \cdot L}$ constant implies $g_K - (g_A + g_L) \approx 0$. That is, $g_K - g_L \approx g_A$. Summing up, $g_{K/L} \approx g_A$.

- Output per person grows at the same rate as technology: techonological progress offsets the diminishing returns to capital.

Summary of results

- Unique steady state.
- The steady state is asymptotically stable and comparative statics is simple.
- There is no sustained growth: growth occurs only when the economy shifts from one steady state to another with a higher $\bar{k}$.
- Sustained growth requires sustained technological change.
- Long-term growth not affected by $\mathbb{s}$ (it only affects the levels of Y, K, and C).

Convergence concepts

- Absolute convergence: poorer economies grow faster regardless of their characteristics.
- Conditional convergence: poorer economies grow faster if they have the same technology and fundamental variables for capital accumulation ($\mathbb{s}$, δ, n) as richer economies.
- Conditional convergence occurs in the SS model: economies with the same technology and fundamental variables converge to the same steady state (similar countries converge).

- Since marginal productivities are decreasing, growth cannot last. In the SS model, (per capita) growth occurs only while the economy converges to the steady state. Once there, per capita variables remain stagnant.

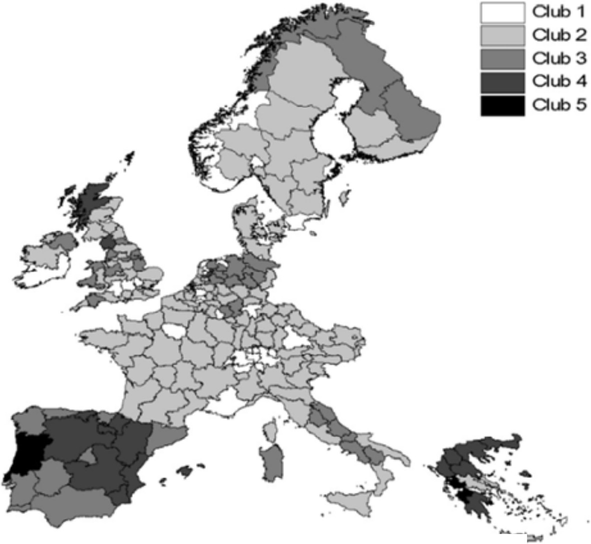
- Strong conditional convergence: convergence conditional on having the same determinants of capital accumulation: $\mathbb{s}$, δ, and n. Relies on the presumption that any technology is eventually available to all economies. What separates economies is capital accumulation not technology.

- Convergence in terms of rates: differences in growth rates (mainly, y) tend to vanish. Does not imply convergence in the level of y.

- There is convergence in rates in the SS model between economies with the same technology: the same rate of technological change yields the same growth rate of output per capita (convergence club = those who innovate).

- The empirically observed convergence has been lower than predicted by the SS model. Convergence: OECD, Asian tigers. Divergence: Africa. Great divergence: 1870-1990.

Convergence clubs, Europe (1990-2005)



http://www.ub.edu/sea2009.com/Papers/129.p

Convergence speed

- The SS model displays a negative correlation between α and the speed of convergence.
- The lower α, the more decreasing returns to capital are. This means that, to converge, capital should accumulate faster.
- With more capital, rates of returns are lower. Capital migrates to poorer economies, where rates of return are higher.
- Through diffusion of knowledge, poorer economies can adopt technological advances developed in richer economies.

(Hirofumi) Uzawa's theorem

- A neoclassical growth model is the SS model except that $\mathbb{s}$ need not be exogenous. The corresponding production function is called neoclassical.

➡ *If a neoclassical growth model has a steady state with constant (non-zero) factors shares* $\frac{F_K \cdot K}{Y}$ *and* $\frac{F_L \cdot L}{Y}$*, and output per capita* $\frac{Y}{L}$ *grows at a rate* g *at the steady state, then, along the steady-state path, the production function can be written as* $Y = G(K, A \cdot L)$*, where* A *grows at rate* g *and* G *is neoclassical.*

Interpreting Uzawa's theorem

- Rough intepretation: in a neoclassical growth model, technical change must either be labour-augmenting or the production function must be Cobb-Douglas (for which Harrod and Solow-neutrality are equivalent).

- Rough intuition. After dividing $Y = F(K, L)$ by Y, $1 = F(\frac{K}{Y}, \frac{1}{y})$. In a steady state $\frac{K}{Y}$ is constant and y grows at rate g. Since effective labour $A \cdot L$ is falling at rate g, technical change must offset this by being a labour-augmenting technical change.

The SS model with human capital

- V will be written instead of $V(t)$, whereas V' will be written instead of $V(t+1)$.
- With $\alpha + \hat{\alpha} < 1$ and H designating human capital, $Y = K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}$ (technical progress is assumed labour-augmenting).
- $N =$ gross rate of growth of L
- $G =$ gross rate of growth of A
- $\mathbb{s} =$ propensity to accumulate K
- $\hat{\mathbb{s}} =$ propensity to accumulate human capital
- $\delta =$ rate at which K depreciates
- $\hat{\delta} =$ rate at which human capital depreciates

- Per capita variables are defined in effective labour units: $k = \frac{K}{A \cdot L}$, $h = \frac{H}{A \cdot L}$, and $y = \frac{Y}{A \cdot L}$.

$$y = \frac{Y}{A \cdot L} = \frac{K^{\alpha} \cdot H^{\widehat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\widehat{\alpha}}}{A \cdot L} = \frac{K^{\alpha} \cdot H^{\widehat{\alpha}}}{(A \cdot L)^{\alpha+\widehat{\alpha}}} =$$

$$= \frac{K^{\alpha}}{(A \cdot L)^{\alpha}} \cdot \frac{H^{\widehat{\alpha}}}{(A \cdot L)^{\widehat{\alpha}}} = \left(\frac{K}{A \cdot L}\right)^{\alpha} \left(\frac{H}{A \cdot L}\right)^{\widehat{\alpha}} = k^{\alpha} \cdot h^{\widehat{\alpha}}.$$

- As in the SS model, $K' = \mathbb{s} \cdot Y + (1-\delta)K$. After dividing both sides by $A' \cdot L'$,

$$k' = \frac{K'}{A' \cdot L'} = \frac{\mathbb{s} \cdot Y}{A' \cdot L'} + \frac{(1-\delta) \cdot K}{A' \cdot L'} =$$

$$= \mathbb{s} \cdot \frac{K^{\alpha} \cdot H^{\widehat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\widehat{\alpha}}}{(G \cdot A) \cdot (N \cdot L)} + \frac{(1-\delta) \cdot K}{(G \cdot A) \cdot (N \cdot L)} =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha} \cdot H^{\widehat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\widehat{\alpha}}}{A \cdot L} + \frac{1-\delta}{G \cdot N} \cdot \frac{K}{A \cdot L} =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha} \cdot H^{\hat{\alpha}}}{(A \cdot L)^{\alpha + \hat{\alpha}}} + \left(\frac{1 - \delta}{G \cdot N}\right) \cdot k =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha}}{(A \cdot L)^{\alpha}} \frac{H^{\hat{\alpha}}}{(A \cdot L)^{\hat{\alpha}}} + \frac{1 - \delta}{G \cdot N} \cdot k =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} + \frac{1 - \delta}{G \cdot N} \cdot k = \frac{\mathbb{s}}{G \cdot N} \cdot y + \frac{1 - \delta}{G \cdot N} \cdot k \,.$$

- By substracting k from both sides,

$$\Delta k = \frac{\mathbb{s}}{G \cdot N} \cdot y - \frac{\delta + G \cdot N - 1}{G \cdot N} \cdot k \,.$$

- This equation coincides with the one in 37 ($G = 1 + a$ and $N = 1 + n$). For $\Delta k = 0$ it must be that $\mathbb{s} \cdot y = (\delta + G \cdot N - 1) \cdot k$. That is,

$$\mathbb{s} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} = (\delta + G \cdot N - 1) \cdot k$$

- Solving for h,

$$h(t) = \left(\frac{\delta + GN - 1}{\mathbb{s}}\right)^{1/\hat{\alpha}} \cdot k(t)^{(1-\alpha)/\hat{\alpha}}$$

- The above equation represents the condition $\Delta k(t) = 0$. On the other hand, starting with

$$H' = \hat{\mathbb{s}} \cdot Y + (1 - \hat{\delta})H$$

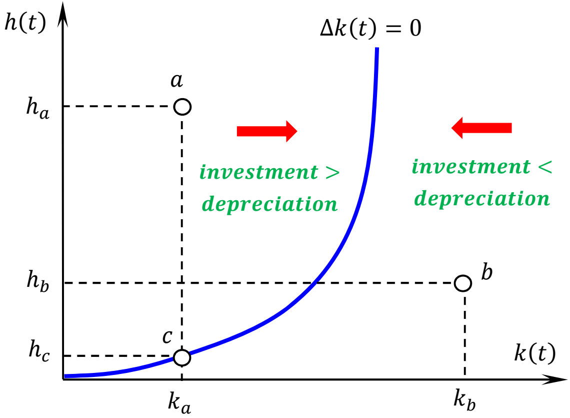
a similar procedure leads to

$$\Delta h = \frac{\hat{\mathbb{s}}}{G \cdot N} \cdot y - \frac{\hat{\delta} + G \cdot N - 1}{G \cdot N} \cdot h\,.$$

- For $\Delta h = 0$ it must be that $\hat{\mathbb{s}} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} = (\hat{\delta} + G \cdot N - 1) \cdot h$. Solving for h, the condition representing $\Delta h(t) = 0$ is

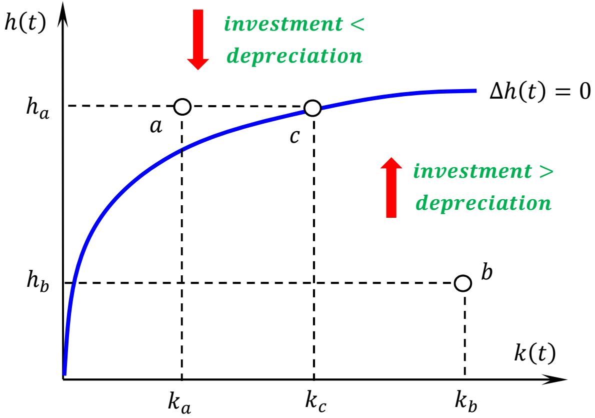
$$h(t) = \left(\frac{\hat{\mathbb{s}}}{\hat{\delta} + GN - 1}\right)^{1/(1-\hat{\alpha})} \cdot k(t)^{\alpha/(1-\hat{\alpha})}\,.$$

Graphical representation of $\Delta k(t) = 0$



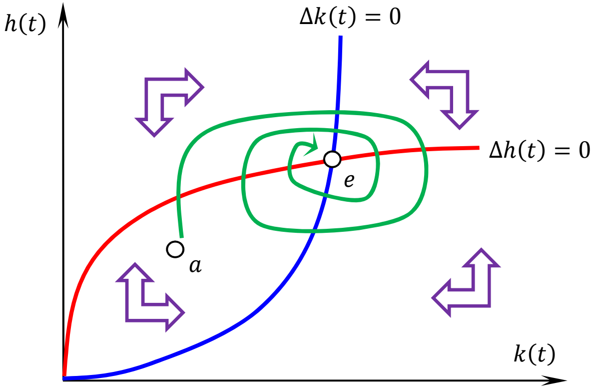
- At points like a, investment in k is higher than depreciation of k, so k increases.
- Investment is higher than depreciation at a because, given k_a, it is enough to have per capita human capital equal to h_c for investment to equal depreciacion (that is, for $\Delta k(t) = 0$ to hold).
- As $h_b > h_c$, there is a human capital excess creating too much output, which generates too much investment (in comparison with the depreciation corresponding to k_a).
- Similarly, k decreases at points to the right of the curve $\Delta k(t) = 0$ (like point b).

Graphical representation of $\Delta h(t) = 0$



- At points like a, investment in h is smaller than depreciation of h, so h decreases.
- Investment is smaller than depreciation at a because, given h_a, it is necessary to have capital stock per capita equal to k_c for investment to equal depreciacion (for $\Delta h(t) = 0$ to hold).
- Since $k_a < k_c$, there is a capital shortage creating an output gap that generates insufficient investment in h (in comparison with the depreciation corresponding to h_a).
- Analogously, h increases at points below the curve $\Delta h(t) = 0$ (like point b).

Solution of the model: $\Delta \boldsymbol{k}(\boldsymbol{t}) = \Delta \boldsymbol{k}(\boldsymbol{t}) = \mathbf{0}$



Dynamics of k & h ensure convergence to e.

The Harrod-Domar model

- Precedes the SS model. It is like the SS model with a non-neoclassical production function.

- $Y = F(K, L) = min\{A \cdot K, B \cdot L\}$, where A, $B > 0$ are fixed coefficients.

- If <u>K is the limiting factor</u> (that is, $A \cdot K < B \cdot L$.), then $Y = A \cdot K$. Hence, $y = A \cdot k = f(k)$.

- Suppose L does not grow. Then, as in the SS model, $\Delta k' = \mathbb{s} \cdot f(k) - \delta \cdot k$. Therefore,

$$g_k = \frac{\Delta k}{k} = \mathbb{s} \cdot \frac{f(k)}{k} - \delta = \mathbb{s} \cdot A - \delta \, .$$

- This means that capital per capita may accumulate at a positive rate if $\mathbb{s} \cdot A > \delta$.

- Since $y = A \cdot k$, $g_y = g_k$. All in all, there is no steady state for the economy: there can be sustained growth of y.

- Growth can be permanent; in the SS model, growth is temporary (it is a by-product of convergence to a steady state and, to be sustained, must be exogenously induced).

- With population growing at net rate n,

$$g_y = g_k = \frac{\mathbb{s}}{1+n} \cdot A - \frac{\delta + n}{1+n} \,.$$

Harrod-Domar model: a shortcoming

- The HD model cannot account for a sustained growth in output per capita Y/L.
- Suppose output Y grows at (net) rate g and population L at rate $n > 0$. Suppose as well that Y/L grows. This requires $g - n > 0$.
- Given that $Y = A \cdot K$, K accumulates at rate g. Accordingly, K/L grows at rate $g - n > 0$.
- That K/L grows at a positive rate means that, eventually, K will no longer be the limiting factor (K/L will exceed B/A).

- That makes L the limiting factor, so $Y = B \cdot L$. In this case, $g = n$ and Y/L cannot grow.

- If L is the limiting factor,

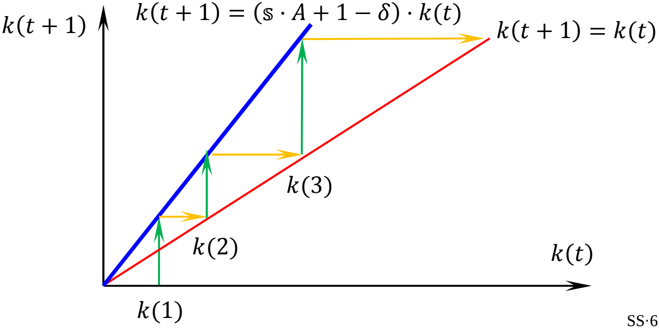
$$g_k = \mathbb{s} \cdot B \cdot \frac{1}{k} - \delta.$$

- When K is the limiting factor, the HD model could account for a (limited) sustained growth of output per capita (long-run growth), whereas the SS model cannot.

- Conversely, the HD model fails to account for convergence among economies, which is a phenomenon the SS model can explain.

The *AK* model /1

- It is like the SS model with $Y = A \cdot K$ always. Can explain sustained long-run growth. In fact, $k(t+1) = (\mathbb{s} \cdot A + 1 - \delta) \cdot k(t)$. Hence, if $\mathbb{s} \cdot A + 1 - \delta > 1$ (that is, $\mathbb{s} \cdot A > \delta$), then k grows forever.



The AK model /2

- All in SS relies on decreasing returns.
- Justification of AK model: the accumulation of capital generates, through learning by doing, technical progress that prevents the productivity of capital from falling.
- Knowledge externalities can then explain sustained long-run growth.
- The AK model fails to account for convergence, as it suggests different growth rates and does not distinguish between capital accumulation and technological progress.

Kaldor's stylized facts of growth

- "Remarkable historical constancies revealed by recent empirical investigation" according to Nicholas Kaldor (1957).

1. Y/L grows (at a roughly constant rate)
2. K/L grows continuously (follows from 1 & 4)
3. Rate of return σ on capital stable
4. K/Y is constant
5. The shares of K and L in output are constant
6. Real wage grows over time (from 2, 4 & 5)
7. Productivity growth differs across countries

More stylized facts of economic growth

- Suggested by Paul Romer (1989).

8. In cross-section, the mean growth rate shows no variation with income per capita levels.
9. The rate of growth of inputs is insufficient to explain the growth of output.
10. Growth in the volume of trade is positively correlated with growth in output.
11. Population growth rates are negatively correlated with the level of income.
12. Skilled and unskilled workers tend to migrate towards high-income countries.

New facts (Jones & Romer, 2009)

1. **Increases in the extent of the market.** Increased flows of goods, ideas, finance, and people — via globalization as well as urbanization — have increased the extent of the market for all workers and consumers.
2. **Accelerating growth.** For thousands of years, growth in both population and per capita GDP has accelerated, rising from virtually zero to the relatively rapid rates observed in the last century.
3. **Variation in modern growth rates.** The variation in the rate of growth of per capita GDP increases with the distance from the technology frontier.
4. **Large income and TFP differences.** Differences in measured inputs explain less than half of the enormous cross country differences in per capita GDP.
5. **Increases in human capital per worker.** Human capital per worker is rising dramatically throughout the world.
6. **Long-run stability of relative wages.** The rising quantity of human capital relative to unskilled labor has not been matched by a sustained decline in its relative price.

http://www.stanford.edu/~chadj/Kaldor200.pdf