Representative-agent (Ramsey) model

- Time is discrete. There is only one good in each period. Expressing variables in per capita terms, at each t, production at t equals consumption at t plus investment at t.

$$y_t = c_t + i_t \tag{1}$$

- Output can only be consumed or saved: $y_t = c_t + s_t$. Therefore, $i_t = s_t$.
- Each period a fraction $0 < \delta < 1$ of capital depreciates. Capital at $t+1$ is investment at t plus the remaining capital from period t.

$$k_{t+1} = i_t + (1-\delta) \cdot k_t \tag{2}$$

- The production function f makes output per capita depend on capital per capita.

$$y_t = f(k_t) \tag{3}$$

- f satisfies the typical properties: $f \geq 0$, $f' > 0$, $f'' < 0$, $\lim_{k_t \to 0} f'(k_t) = \infty$, and $\lim_{k_t \to \infty} f'(k_t) = 0$.

- Combining (1), (2), and (3),

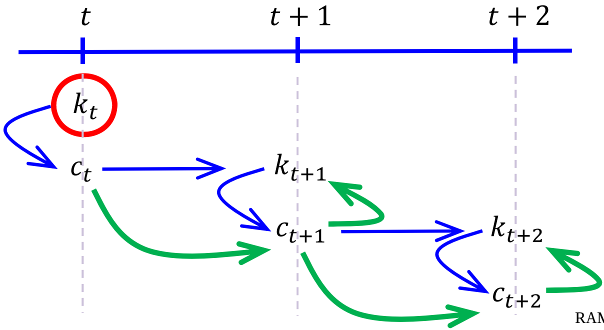
$$f(k_t) = c_t + k_{t+1} - (1-\delta) \cdot k_t \tag{4}$$

or, by defining $\Delta k_{t+1} = k_{t+1} - k_t$,

$$f(k_t) = c_t + \Delta k_{t+1} + \delta \cdot k_t. \tag{5}$$

Dynamic constraint on the economy

- (5) defines the dynamic constraint the economy faces. Interpretation 1: given k_t, c_t and k_{t+1} are determined; given k_{t+1}, c_{t+1} and k_{t+2} are determined… Interpretation 2: given k_t, decision is over c_t, c_{t+1}, c_{t+2}…



Consumption maximization (golden rule)

- There is a representative agent. If population is constant, then variables can be seen as per capita variables (c_t would be what the agent consumes in period t).

- Suppose the aim of the agent is to maximize consumption each period (no discounting).

- The problem can be solved by considering the steady state (the long run of the economy). Let c and k be the steady state values.

- From (5), $f(k) = c + \delta \cdot k$; i.e., $c = f(k) - \delta \cdot k$.

- This is the familiar idea that steady-state consumption is the output that remains once taken the output necessary to replace the lost capital, so that capital remains constant.

- The first-order condition to maximize c is $\frac{\partial c}{\partial k} = 0$; that is, $f'(k) = \delta$. Since $f'' < 0$, the second-order condition $\left(\frac{\partial^2 c}{\partial k^2} < 0\right)$ holds.

- $f'(k) = \delta$ says that the marginal product of capital equals its depreciation rate. This solution is known as "the golden rule". If $f'(k) < \delta$, c can be increased by rising c. If $f'(k) > \delta$, c can be increased by lowering k.

Shocks and the golden rule

- Let (c_G, k_G) be the golden rule solution. Suppose capital is exogenously reduced to $k < k_G$ but the agent tries to maintain c_G.

- Then $c_G = f(k_G) - \delta \cdot k_G$ and $c = f(k) - \delta \cdot k - \Delta k$. If $c_G = c$, then $f(k_G) - \delta \cdot k_G = f(k) - \delta \cdot k - \Delta k$. Solving for Δk,

$$\Delta k = (f(k) - \delta \cdot k) - (f(k_G) - \delta \cdot k_G).$$

- As (c_G, k_G) is the golden rule solution $f(k_G) - \delta \cdot k_G > f(k) - \delta \cdot k$. In sum, $\Delta k < 0$.

- With less capital, future output would be smaller. The attempt to keep c_G will further decrease the stock of capital, making the consumption level c_G eventually untenable.

- Lesson: "too much" consumption sooner or later exhausts the capital stock, so the economy will be unable to sustain that consumption level.

- Solution to the negative shock on k: divert consumption temporarily to rebuild the capital stock. Once k_G is restored, c can be increased to reach level c_G.

Utility maximization: problem

- If consumption in different periods is valued differently, the agent may choose to <u>maximize</u> the present value of the infinite sequence of consumption $(c_0, c_1, c_2, \ldots)$ or, given a utility function u common for each t, <u>the present value</u> of $(u(c_0), u(c_1), u(c_2), \ldots)$.

$$max_{\{c_t,\, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to } \; c_t + k_{t+1} = f(k_t) + (1 - \delta) \cdot k_t$$

- Assumptions on u: $u \geq 0$, $u' > 0$, and $u'' < 0$. Parameter $\beta \in (0, 1)$ is the discount factor.

Utility maximization: solution

- Using the method of Lagrange multipliers, define the Lagrangian as

$$\mathcal{L}_t = \sum_{t=0}^{\infty}[\beta^t \cdot u(c_t) + \lambda_t(f(k_t) + (1-\delta) \cdot k_t - c_t - k_{t+1})]$$

which is maximized w.r.t. c_t, k_{t+1}, and λ_t ($\mathcal{L}_t$ is not maximized w.r.t. k_t because k_t is known at t).

First-order conditions

$$0 = \frac{\partial \mathcal{L}_t}{\partial c_t} = \beta^t \cdot u'(c_t) - \lambda_t$$

$$0 = \frac{\partial \mathcal{L}_t}{\partial k_{t+1}} = \lambda_{t+1}(f'(k_{t+1}) + 1 - \delta) - \lambda_t$$

$$0 = \frac{\partial \mathcal{L}_t}{\partial \lambda_t} = f(k_t) + (1-\delta) \cdot k_t - c_t - k_{t+1}$$

Transversality condition (TC)

$$\lim_{t\to\infty} \beta^t \cdot u'(c_t) \cdot k_{t+1} = 0$$

- To interpret TC, suppose t is the last period.
- If $k_{t+1} > 0$ (some capital is left at the last period), then $u'(c_t) = 0$: consuming that capital should have no impact on utility.
- If $u'(c_t) > 0$, then it cannot be that some capital is saved for the next (non-existent) period, because utility would be increased by consuming that capital now. Therefore, it must be that $k_{t+1} = 0$.

Euler equation

- From the first FOC, $\lambda_t = \beta^t \cdot u'(c_t)$ and $\lambda_{t+1} = \beta^{t+1} \cdot u'(c_{t+1})$. Substituting for λ_t and λ_{t-1} in the second FOC,

$$\beta^{t+1} \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + 1 - \delta] = \beta^t \cdot u'(c_t)\,.$$

- The result is the so-called Euler equation:

$$\beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + 1 - \delta] = u'(c_t)\,. \quad (6)$$

- Interpretation. How much additional c_{t+1} can be obtained by just reducing c_t while leaving total utility (and everything beyond period $t + 1$) constant?

- Since periods after $t+1$ are unaffected, attention can be restricted to $u(c_t) + \beta \cdot u(c_{t+1})$, which must remain constant. Taking the total differential,

$$0 = du(c_t) + d[\beta \cdot u(c_{t+1})] = du(c_t) + \beta \cdot du(c_{t+1}) =$$
$$= u'(c_t) \cdot dc_t + \beta \cdot u'(c_{t+1}) \cdot dc_{t+1} .$$

- In sum,

$$-\frac{dc_{t+1}}{dc_t} = \frac{u'(c_t)}{\beta \cdot u'(c_{t+1})} . \qquad (7)$$

- This is nothing else but the MRS. The resource constraints at t and $t+1$ must hold, so

$$dc_t + dk_{t+1} = df(k_t) + (1-\delta) \cdot dk_t$$
$$dc_{t+1} + dk_{t+2} = df(k_{t+1}) + (1-\delta) \cdot dk_{t+1}.$$

- That is,

$$dc_t + dk_{t+1} = f'(k_t) \cdot dk_t + (1-\delta) \cdot dk_t$$
$$dc_{t+1} + dk_{t+2} = f'(k_{t+1}) \cdot dk_{t+1} + (1-\delta) \cdot dk_{t+1}$$

- Since k_t is given at t, $dk_t = 0$. The first equation then becomes $dk_{t+1} = -dc_t$: the additional capital at $t+1$ comes from the consumption cut at t.

- By assumption, $dk_{t+2} = 0$. Given $dk_{t+1} = -dc_t$, the second equation is equivalent to

$$dc_{t+1} = -f'(k_{t+1}) \cdot dc_t - (1-\delta) \cdot dc_t$$

or

$$-\frac{dc_{t+1}}{dc_t} = f'(k_{t+1}) + \ (1-\delta).$$

- From this and (7) the Euler equation follows.

- Interpretation. The output dc_t not consumed at t yields a utility loss at t of $|u'(c_t) \cdot dc_t|$. This output is invested at $t+1$, as dk_{t+1}, to increase output at $t+1$.

- The additional output $|f'(k_{t+1}) \cdot dc_t|$ and the undepreciated part $(1-\delta) \cdot dk_{t+1} = |(1-\delta) \cdot dc_t|$ of the extra capital are consumed at $t+1$. All in all,

$$dc_{t+1} = [f'(k_{t+1}) + (1-\delta)] \cdot |dc_t|.$$

- The discounted utility of dc_{t+1} is

$$\beta \cdot u'(c_{t+1}) \cdot dc_{t+1} =$$
$$\beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + (1 - \delta)] \cdot |dc_t|.$$

- But to keep utility constant, the utility $\beta \cdot u'(c_{t+1}) \cdot dc_{t+1}$ gained at $t + 1$ must equal the utility $u'(c_t) \cdot |dc_t|$ lost at t. As a result,

$$u'(c_t) \cdot |dc_t| = \beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + (1 - \delta)] \cdot |dc_t|$$

  which is the Euler equation once the common term $|dc_t|$ is cancelled out.

Steady state solution

- For steady-state values c and k, the Euler equation can be written as

$$\beta \cdot u'(c) \cdot [f'(k) + 1 - \delta] = u'(c)$$

so

$$f'(k) = \delta + \frac{1}{\beta} - 1\,.$$

- The golden rule solution is $f'(k_G) = \delta$. Since $\frac{1}{\beta} - 1 > 0$, $f'(k) > f'(k_G)$. As $f'' < 0$, $k < k_G$. There is less capital than under the golden rule because now future utility is discounted at a rate $\frac{1}{\beta} - 1$. Moreover, $k < k_G$ yields $c < c_G$: discounting lowers consumption.

Dynamic analysis

- The dynamic analysis relies on the two equations giving the solution at each t: Euler equation (6) and the resource constraint (5).

$$\beta \cdot \frac{u'(c_{t+1})}{u'(c_t)} \cdot [f'(k_{t+1}) + 1 - \delta] = 1$$

$$\Delta k_{t+1} = f(k_t) - c_t - \delta \cdot k_t \qquad (8)$$

- Linearizing the Euler equation by taking a Taylor series expansion of $u'(c_{t+1})$ around c_t,

$$u'(c_{t+1}) \approx u'(c_t) + \Delta c_{t+1} \cdot u''(c_t)$$

or

$$\frac{u'(c_{t+1})}{u'(c_t)} \approx 1 + \Delta c_{t+1} \cdot \frac{u''(c_t)}{u'(c_t)}.$$

- Inserting the previous approximation into the Euler equation yields (9), where $\frac{u''}{u'} < 0$.

$$\Delta c_{t+1} = \frac{u''(c_t)}{u'(c_t)} \left(\frac{1}{\beta \cdot [f'(k_{t+1}) + 1 - \delta]} - 1 \right) \quad (9)$$

- Equations (8) and (9) establish the changes in the capital stock and consumption.

- Let c and k be the steady-state values (the solutions of (8) and (9) if $\Delta k_{t+1} = \Delta c_{t+1} = 0$).

- If $k_{t+1} < k$, then $f'(k_{t+1}) > f'(k)$. Hence,

$$\beta \cdot [f'(k_{t+1}) + 1 - \delta] > \beta \cdot [f'(k) + 1 - \delta].$$

- As shown in RAM16, $f'(k) = \delta + \frac{1}{\beta} - 1$. Thus, $\beta \cdot [f'(k) + 1 - \delta] = 1$.

- Consequently, $\beta \cdot [f'(k_{t+1}) + 1 - \delta] > 1$ and, in (9), $\frac{1}{\beta \cdot [f'(k_{t+1}) + 1 - \delta]} < 1$. Since $\frac{u''}{u'} < 0$, the final conclusion is that

$$k_{t+1} < k \Rightarrow \Delta c_{t+1} > 0 \,.$$

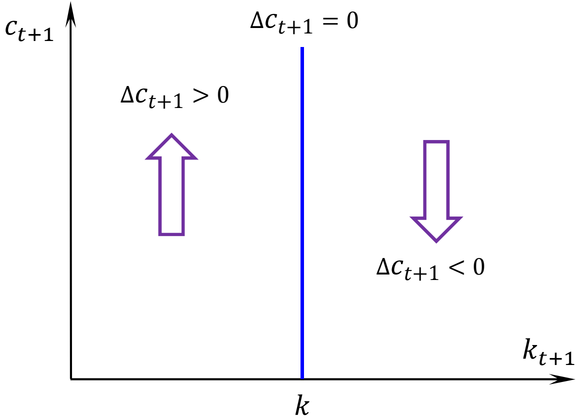
- A similar reasoning proves that

$$k_{t+1} > k \Rightarrow \Delta c_{t+1} < 0$$

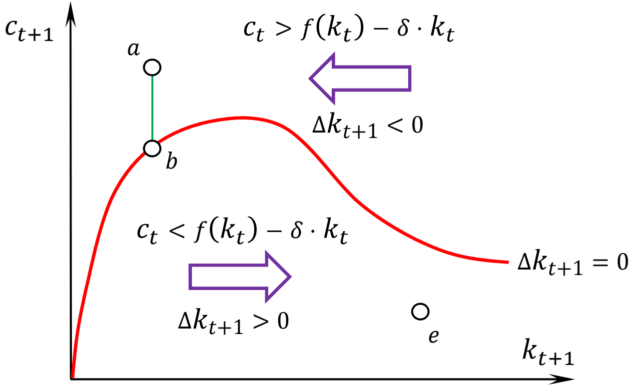
$$k_{t+1} = k \Rightarrow \Delta c_{t+1} < 0.$$

- This consumption dynamics is represented in the next figure: for capital stock to the left of the steady-state value k, consumption increases; for stock to the right of k, consumption decreases.

Consumption dynamics



Capital dynamics



- The previous figure shows the capital dynamics following (8). Clearly,

$$\Delta k_{t+1} > 0 \quad \Leftrightarrow f(k_t) - \delta \cdot k_t > -c_t \,.$$

- Above the curve $\Delta k_{t+1} = 0$, consumption is higher that the steady-state consumption, so capital must decumulate.

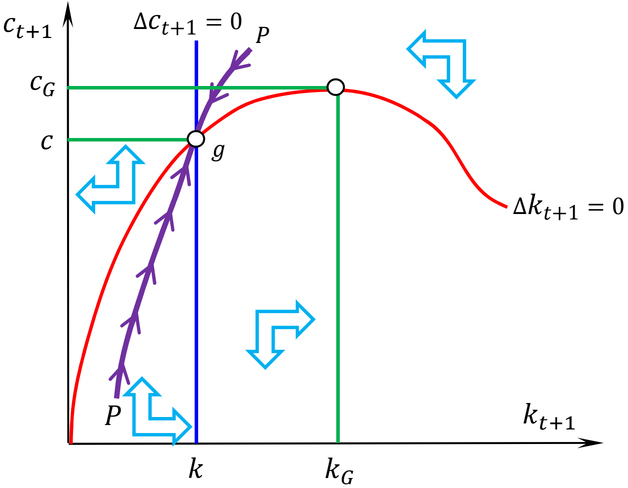
- At point a, consumption exceeds the level (given by b) compatible with the steady state (with $\Delta k_{t+1} = 0$). Capital has to decrease to compensate excessive consumption.

- Below the curve $\Delta k_{t+1} = 0$, consumption allows capital to acumulate.

Phase diagram

- When the two preceding figures are put together (see RAM24), the steady-state solution can be identified as the intersection g of the curves $\Delta k_{t+1} = 0$ and $\Delta c_{t+1} = 0$. The arrows show the dynamics of k_{t+1} and c_{t+1}.

- The curve PP (the saddlepath or stable manifold) indicates the only states that are attainable (PP may change when some parameter of the model is modified). If the economy were outside PP, the dynamics guarantees that the steady state is never reached.



c_{t+1}
$\Delta c_{t+1} = 0$
P
c_G
c
g
$\Delta k_{t+1} = 0$
P
k_{t+1}
k
k_G