Axiomatic characterizations of the majority rule (two alternatives)

1. The characterization by Kenneth May[1] (fixed population)

1.1. Definitions

N	finite set of n individuals (N can be defined as $\{1, \dots , n\}$)
$\{x, y\}$	set of alternatives (possible decisions)
preference over $\{x, y\}$	represented by a number from the set $\{-1, 0, 1\}$
1	represents the preference in which x is preferred to y
-1	represents the preference in which y is preferred to x
0	represents the preference in which x and y are indifferent
preference profile p	a function $p : N \to \{-1, 0, 1\}$ assigning a preference over $\{x, y\}$ to each individual
p_i	preference $p(i)$ of individual I in preference profile p
p_{-i}	preferences held by individuals other than i in profile p
P	set of all preference profiles
social welfare function f	a mapping $f : P \to \{-1, 0, 1\}$ assigning a collective preference $f(p)$ to each preference profile $p \in P$
(relative) majority rule μ	social welfare function that safisfies, for all $p \in P$: (i) if $\sum_{i\in I} p_i > 0$, then $\mu(p) = 1$; (ii) if $\sum_{i\in I} p_i < 0$, then $\mu(p) = -1$; and (iii) if $\sum_{i\in I} p_i = 0$, then $\mu(p) = 0$

1.2. Axioms

NEU. *Neutrality*
For all $p \in P$, $f(-p_1, \dots , -p_n) = -f(p_1, \dots , p_n)$.

ANO. *Anonymity*
For all $p \in P$ and $q \in P$, if, for all $a \in \{-1, 1\}$, $|\{i \in I\colon p_i = a\}| = |\{i \in I\colon q_i = a\}|$, then $f(p) = f(q)$, where $|S|$ stands for the cardinality of the finite set S. [Equivalently, $f(q) = f(p)$ if q is obtained from $p \in P$ by permuting the preferences of two individuals.]

PR. *Positive responsiveness*
For all $p \in P$ and $a \in \{0, 1\}$, if $f(p) \in \{0, 1\}$ and $a > p_i$, then $f(p_{-i}, a^i) = 1$.

[1] May, K. O. (1952): "A set of independent, necessary and sufficient conditions for simple majority decision", Econometrica 20, 680–684.

NEU asserts that if all the individuals' preferences are reversed, then the corresponding collective preference is also reversed.

ANO states that the collective preference does not depend on the order in which the individuals' preferences are collected: the outcome is not affected by any two individuals having their preferences exchanged.

PR is a monotonicity property. Combined with NEU, PR holds that if the collectively most preferred alternative is given more support, then it remains the collectively most preferred alternative. It also holds that if the society is indifferent, then giving more support to an alternative transforms indifference into preference for that alternative.

1.3. Characterization (fixed society case). *A social welfare function f satisfies NEU, ANO, and PR if and only if f is the majority rule* μ.

2. A characterization by Yongsheng Xu and Zhen Zhong[2] (variable population)

2.1. Definitions

N	set of n individuals (N can be identified with the set $\mathbb{N}$ of positive integers)
society	a finite non-empty subset of N
$\{x, y\}$	set of alternatives (possible decisions)
preference over $\{x, y\}$	represented by a number from the set $\{-1, 0, 1\}$
1	represents the preference in which x is preferred to y
-1	represents the preference in which y is preferred to x
0	represents the preference in which x and y are indifferent
preference profile p_I for society I	a function $p_I : I \to \{-1, 0, 1\}$ assigning a preference over $\{x, y\}$ to each individual in society I

An alternative interpretation is that the preference profile p_I represents an election: $p_I(i) = 1$ means that individual i votes for alternative (or candidate) x, $p_I(i) = -1$ means that i votes for y, and $p_I(i) = 0$ means that i abstains.

[2] Xu, Y. and Zhong, Z. (2010): "Single profile of preferences with variable societies: A characterization of simple majority rule", Economics Letters 107, 119–121. Though the result keeps a preference profile fixed and allows societies to vary, it is valid for the case in which both societies and preferences change.

p_i	abbreviates $p_I(i)$ with a given society I
a^I	for $a \in \{-1, 0, 1\}$ and society, abbreviates the preference profile p_I for I such that, for all $i \in I$, $p_i = a$ (when $I = \{i\}$, a^i is written instead of $a^{\{i\}}$)
$p_{I\cup J}$	given profiles p_I and p_J of disjoint societies I and J, $p_{I\cup J}$ is the profile for $I \cup J$ such that $p_{I\cup J}(i) = p_I(i)$ if $i \in I$ and $p_{I\cup J}(i) = p_J(i)$ if $i \in J$
restriction of p_I to $J \subset I$	preference profile p_J for J such, for all $i \in J$, $p_J(i) = p_I(i)$
P	set of all preference profiles for all societies
social welfare function f	a mapping $f : P \to \{-1, 0, 1\}$ associating a collective preference $f(p_I)$ with each $p_I \in P$
(relative) majority rule μ	social welfare function that safisfies, for all $p_I \in P$: (i) if $\sum_{i\in I} p_i > 0$, then $\mu(p_I) = 1$; (ii) if $\sum_{i\in I} p_i < 0$, then $\mu(p_I) = -1$; and (iii) if $\sum_{i\in I} p_i = 0$, then $\mu(p_I) = 0$

2.2. Axioms

UNA. *Unanimity*
For every society I and each $a \in \{-1, 0 , 1\}$, $f(a^I) = a$.

SET. *Simple equal treatment*
For all $i \in I$ and $j \in I\backslash\{i\}$, $f(1^i, -1^j) = 0$.

IUC. *Independence of an unconcerned coalition*
For all $p_I \in P$ and $p_J \in P$ such that $I \cap J = \varnothing$, if $f(p_I) = 0$, then $f(p_{I\cup J}) = f(p_J)$.

SD states that, in societies in which all the individuals hold the same preference, that preference corresponds to the collective preference.

SET requires indifference to be the result of having two individuals with opposite preferences.

IUC says that a society J joining an indifferent society I determines the preference of the aggregate society $I \cup J$.

2.3. Characterization (variable society case). *A social welfare function f satisfies UNA, SET, and IUC if and only if f is the majority rule μ.*

3. Another characterization for the variable population case[3]

3.1. Definitions. The same as in section 2.

3.2. Axioms

MON. *Monotonicity*
For all $p_I \in P$ and $p_J \in P$ such that $I \cap J = \varnothing$, if $f(p_I) = f(p_J)$, then $f(p_{I \cup J}) = f(p_I)$.

EFF. *Efficiency*
For all $p_I \in P$ and $i \in I$, if $f(p_{I \setminus \{i\}}) = 0$ or $I \setminus \{i\} = \varnothing$, then $f(p_I) = p_i$.

CON. *Continuity*
For all $p_I \in P$ and $i \in I$, if $f(p_I) \neq 0$, then $f(p_{I \setminus \{i\}}) \neq -f(p_I)$.

MON asserts that the common collective preference of two disjoint societies is preserved by merging those societies.

EFF is a Pareto optimality condition: if an individual joins an indifferent society, then the individual determines the preference of the new society.

According to CON, if a society is not indifferent, then the removal of an individual cannot reverse the collective preference.

3.3. Characterization (variable society case). *A social welfare function f satisfies MON, EFF, and CON if and only if f is the majority rule μ.*

4. A parallel characterization of the relative and the absolute majority rules

4.1. Definitions. The same as in section 2 plus the following ones.

P_r set of all preference profiles for societies with exactly $r \in \mathbb{N}$ members

For $a \in \{-1, 0, 1\}$, define $n_a(p_I) = |\{i \in I: p_i = a\}|$ to be the number of individuals in society I having preference a in preference profile p_I for I.

[3] "Monotonicity + efficiency + continuity = majority", Mathematical Social Sciences 60, 149–153 (2010).

The <u>relative majority</u> rule is the social welfare function μ such that, for all $p_I \in P$: (i) $n_1(p_I) > n_{-1}(p_I)$ implies $\mu(p_I) = 1$; (ii) $n_1(p_I) < n_{-1}(p_I)$ implies $\mu(p_I) = -1$; and (iii) $n_1(p_I) = n_{-1}(p_I)$ implies $\mu(p_I) = 0$.

The <u>absolute majority</u> rule is the social welfare function α such that, for all $p_I \in P$: (i) $n_1(p_I) > n_{-1}(p_I) + n_0(p_I)$ implies $\alpha(p_I) = 1$; (ii) $n_{-1}(p_I) > n_1(p_I) + n_0(p_I)$ implies $\alpha(p_I) = -1$; and (iii) otherwise, $\alpha(p_I) = 0$.

The <u>unanimity</u> rule is the social welfare function υ such that, for all $p_I \in P$: (i) $n_{-1}(p_I) + n_0(p_I) = 0$ implies $\upsilon(p_I) = 1$; (ii) $n_1(p_I) + n_0(p_I) = 0$ implies $\upsilon(p_I) = -1$; and (iii) otherwise, $\upsilon(p_I) = 0$.

For all $p_I \in P \backslash P_1$ and $a \in \{-1, 1\}$, a <u>dominates</u> $-a$ given social welfare function f and preference profile p_I if:

(i) for each non-empty $J \subset I$, $a \in \{f(p_J), f(p_{I\backslash J})\}$; and
(ii) for some non-empty $J \subset I$, $-a \notin \{f(p_J), f(p_{I\backslash J})\}$.

In other words, strict preference a dominates the opposite strict preference $-a$ if, for every partition of society I into two subsocieties, a is supported by at least one of the two subsocieties and, for some binary partition, none of the subsocieties supports $-a$.

4.2. Axioms

A1. For all $p_I \in P_1$, $f(p_I) = \upsilon(p_I)$
A1′. For all $p_I \in P_1 \cup P_2$, $f(p_I) = \upsilon(p_I)$.

A2. For all $p_I \in P \backslash P_1$ and $a \in \{-1, 1\}$, $f(p_I) = a$ if and only if a dominates $-a$ given f and p_I.
A2′. For all $p_I \in P \backslash (P_1 \cup P_2)$ and $a \in \{-1, 1\}$, $f(p_I) = a$ if and only if a dominates $-a$ given f and p_I.

4.3. Characterizations

• *A social welfare function f satisfies A1 and A2 if and only if f is the relative majority rule μ.*

• *A social welfare function f satisfies A1′ and A2′ if and only if f is the absolute majority rule α.*

Majority rule with domain restrictions: Theorems by Duncan Black[4]

1. Problem. The majority rule defined for more than two alternatives may generate preference cycles, which violate the transitivity of preferences. Is there a restriction on the type of allowed preferences making the majority rule immune to that flaw?

2. Definitions

N	finite set of individuals
A	finite set of alternatives
p	preference profile assigning a strict preference p_i (indifference between alternatives not allowed) to each individual
majority relation M_p	binary relation on A such that $a\ M_p\ b$ if the number of individuals preferring a to b in p is equal or greater than the number of individuals preferring b to a in p
Condorcet winner in p	any alternative a such that, for all $b \in A$, $a\ M_p\ b$
single-peaked preferences	the preferences in preference profile p are single-peaked if there is a linear ordering on A, represented by a betweenness binary relation $B(x, y, z)$ on A [$B(x, y, z)$ meaning that y is between x and z], such that, for each individual i, if x is preferred to y in p_i and $B(x, y, z)$, then y is preferred to z in p_i.

3. Results

• **Black's theorem (version 1).** If N has an odd number of members and the preferences in p are single-peaked, then the majority binary relation M_p is transitive. Hence, with finite N, for every preference profile p, there exists a unique Condorcet winner in p.

• **Black's theorem (version 2).** If N has an even number of members and the preferences in p are single-peaked, then the majority binary relation M_p is quasi-transitive (the strict component of M_p is transitive). Thus, with finite N, for every preference profile p, there exists some (not necessarily unique) Condorcet winner in p.

[4] Black, D. (1948): "On the rationale of group decision making", Journal of Political Economy 56, 23–34.