

Nth Arrovian impossibility

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Abstract

A dictatorship result for preference aggregation problems is obtained, without any collective rationality assumption, from a resoluteness requirement and a strengthening of the independence condition in Arrow's theorem. Specifically, for resolute aggregation rules that generate complete binary relations, having a strong dictator (an individual who imposes both his strict preference and his indifference) is equivalent to satisfying the strengthened independence condition. This result is a further illustration of the close relationship between independence conditions and dictatorship.

Keywords: Arrow's theorem, independence of irrelevant alternatives, collective rationality, dictatorship.

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1. Introduction

Starting from Arrow's (1963, p. 97) theorem, several dictatorship, weak dictatorship and veto power results have been obtained by retaining the independence of irrelevant alternatives condition (IIA) and assuming collective rationality properties weaker than transitivity. Examples of these results can be found in Mas-Colell and Sonnenschein (1972), Blau and Deb (1977), Sen (1977), Blair and Pollak (1979, 1982), Kelsey (1985) and Denicolò (1987).

The quantity and variety of such results suggest that collective rationality assumptions are somewhat contingent postulates in Arrow's dictatorship theorem and that the concentration of power in preference aggregation is more likely to be due to independence type assumptions; see Aleskerov (1999) for a comprehensive analysis of aggregation under independence conditions. In this respect, Saari (1998) argues that IIA should be found guilty of being an inappropriate requirement when the preferences being aggregated are transitive.

This note presents a result illustrating how dispensable collective rationality axioms are in the presence of a sufficiently strong independence condition. The aggregation rules considered transform profiles of transitive preferences into complete binary relations, without postulating any collective rationality property. It is assumed that such rules are resolute in the sense that, for every two different alternatives, it is not the case that they are always declared socially indifferent. A strengthening of IIA is suggested, based on weakening the property that IIA preserves in the preference aggregation and next strengthening the new property (which refers to arbitrary pairs of alternatives and not just a pair and its converse). A strong dictator is an individual whose preference always coincides with the social preference. The result is that resolute aggregation rules have a strong dictator if, and only if, the strengthened independence condition holds. This result can then be viewed as a further challenge to the appropriateness of independence type conditions in preference aggregation.

2. Definitions and assumptions

Members of the finite set $N = \{1, 2, \dots, n\}$ designate individuals, the m elements of the finite set A represent alternatives, C is the set of complete binary relations that can be defined on A , $T := \{r \in C: r \text{ is transitive}\}$ and $L := \{r \in T: r \text{ is asymmetric}\}$. For $R \in T^n$, $x \in A$, $y \in A \setminus \{x\}$ and non-empty $G \subset N$: (i) R_i denotes the i th component of R ; (ii) if $\xi =$

(x, y) then $\xi' := (y, x)$; (iii) $D_{xy} := \{(x, y), (y, x)\}$; and (iv) $R \cap D_{xy} = (x^G, y^{-G})$ means that, for all $i \in G$, $R_i \cap D_{xy} = \{(x, y)\}$ and, for all $i \in \setminus G$, $R_i \cap D_{xy} = \{(y, x)\}$.

Definition 2.1. For $f: L^n \rightarrow C$, a non-empty strict subset G of N is: (i) almost decisive for $x \in A$ against $y \in A \setminus \{x\}$ in f if, for all $R \in L^n$ such that $R \cap D_{xy} = (x^G, y^{-G})$, $f(R) \cap D_{xy} = \{(x, y)\}$; (ii) decisive for $x \in A$ against $y \in A \setminus \{x\}$ in f if, for all $R \in L^n$ such that, for all $i \in G$, $R_i \cap D_{xy} = \{(x, y)\}$, $f(R) \cap D_{xy} = \{(x, y)\}$; and (iii) (almost) decisive if, for all $x \in A$ and $y \in A \setminus \{x\}$, G is (almost) decisive for x against y in f .

Definition 2.2. A strong dictator in $f: T^n \rightarrow C$ is an $i \in N$ such that, for all $R \in T^n$, $f(R) = R_i$. That $f: T^n \rightarrow C$ is strongly dictatorial means that there is a strong dictator in f .

In the usual definition of dictatorship, i is a dictator when $\emptyset \neq R_i \cap D_{xy} \neq D_{xy}$ implies $f(R) \cap D_{xy} = R_i \cap D_{xy}$. In the case at hand, dictatorship is strong in the sense that some i imposes not only his strict preference but also his indifference.

A1. For all $x \in A$ and $y \in A \setminus \{x\}$, there is $R \in T^n$ such that $(x, y) \notin f(R)$.

By A1, no $(x, y) \in A^2$ with $x \neq y$ is always selected: there is some preference profile R under which x is not declared at least as socially preferred as y , so that $f(R) \cap D_{xy} \neq D_{xy}$.

A2. Let $R, Q \in T^n$ and $\xi, \zeta \in A^2$. If, for all $i \in N$, $[\{\xi, \zeta\} \subseteq R_i \Leftrightarrow \{\xi, \zeta\} \subseteq Q_i]$ then $[\{\xi, \zeta\} \subseteq f(R) \Leftrightarrow \{\xi, \zeta\} \subseteq f(Q)]$.

A2, like IIA, can be viewed as an attempt to transfer to the social preference a property satisfied by all the individual preferences. Let $r, q \in T$ and $\xi, \zeta \in A^2$. The property preserved in IIA is $\phi(\xi, \xi', r, q) = “(\xi \in r \Leftrightarrow \xi \in q) \wedge (\xi' \in r \Leftrightarrow \xi' \in q)”$, so IIA states that if, for all $i \in N$, $\phi(\xi, \xi', R_i, Q_i)$ then $\phi(\xi, \xi', f(R), f(Q))$. A2 comes from IIA by applying a weakening and a strengthening: $\phi(\xi, \xi', r, q)$ is replaced by the weaker property $\varphi(\xi, \xi', r, q) = “(\xi \in r \wedge \xi' \in r) \Leftrightarrow (\xi \in q \wedge \xi' \in q)”$ and next ξ' is allowed to be an arbitrary pair, so A2 states that if, for all $i \in N$, $\varphi(\xi, \zeta, R_i, Q_i)$ then $\varphi(\xi, \zeta, f(R), f(Q))$. In view of this, it should not be difficult to verify Remark 2.3.

Remark 2.3. A2 implies IIA.

The reader should not be misled: it is not presumed that A2 is a reasonable property or that φ is a property worth preserving. The aim is not to produce an undesirable result from a reasonable assumption but to make evident how an independence condition (so

close in spirit to IIA) is capable of concentrating power to such an extent that a strong dictator emerges.

3. Result

It is assumed in Lemmas 3.1-3.5 that $n \geq 2 < m$ and that $f: T^n \rightarrow C$ satisfies A1 and A2. For $f: T^n \rightarrow C$, define $f_L: L^n \rightarrow C$ to be the restriction of f to L^n ; that is, for all $R \in L^n$, $f_L(R) := f(R)$.

Lemma 3.1. Suppose $R \in L^n$, non-empty $G \subset N$, $x \in A$ and $y \in A \setminus \{x\}$ are such that $R \cap D_{xy} = (x^G, y^{-G})$. Then either $f_L(R) \cap D_{xy} = \{(x, y)\}$ or $f_L(R) \cap D_{xy} = \{(y, x)\}$.

Proof. As $f(R)$ is complete, it is enough to derive a contradiction from the assumption that $f_L(R) \cap D_{xy} = D_{xy}$. Suppose $f_L(R) \cap D_{xy} = D_{xy}$. Let $\alpha = (x, y)$. Step 1: for all $Q \in L^n$, $D_{xy} \subset f(Q)$. Choose $Q \in L^n$. Then, for all $i \in N$, it is false that $\{\alpha, \alpha'\} \subseteq R_i$ and it is false that $\{\alpha, \alpha'\} \subseteq Q_i$. By A2 with $\xi = \alpha$ and $\zeta = \alpha'$, $f_L(Q) \cap D_{xy} = D_{xy}$. Step 2: for all $R \in T^n \setminus L^n$, $\alpha \in f(R)$. Choose $R \in T^n \setminus L^n$. Let $Q \in L^n$ be obtained from R by removing α' from each R_i such that $\{\alpha, \alpha'\} \subseteq R_i$. Hence, for all $i \in N$, $\alpha \in R_i$ if, and only if, $\alpha \in Q_i$. By A2 with $\xi = \zeta = \alpha$, $\alpha \in f(R)$ if, and only if, $\alpha \in f(Q)$. By step 1, $\alpha \in f(Q)$. As a result, $\alpha \in f(R)$. Step 3: for all $R \in T^n$, $\alpha \in f(R)$. This follows from steps 1 and 2. Step 3 contradicts A1. ■

Lemma 3.2. If there are $x \in A$ and $y \in A \setminus \{x\}$ such that $G \subset N$ is almost decisive for x against y in f_L then, for all $x \in A$ and $y \in A \setminus \{x\}$, G is almost decisive for x against y in f_L .

Proof. Suppose $f(R) \cap D_{xy} = \{(x, y)\}$, with $R \cap D_{xy} = (x^G, y^{-G})$. Letting $z \in A \setminus \{x, y\}$, it is enough to show: (i) that $f(Q) \cap D_{xz} = \{(x, z)\}$ when $Q \cap D_{xz} = (x^G, z^{-G})$; and (ii) that $f(S) \cap D_{zy} = \{(z, y)\}$ when $S \cap D_{zy} = (z^G, y^{-G})$. Case (i). Suppose there is $Z \in L^n$ such that $Z \cap D_{xz} = (x^G, z^{-G})$ but, by Lemma 3.1, $f(Z) \cap D_{xz} = \{(z, x)\}$. By Remark 2.3, $Y \in L^n$ such that $Y \cap D_{xy} = (x^G, y^{-G})$ and $Y \cap D_{zy} = (y^G, z^{-G})$ yields $f(Y) \cap (D_{xy} \cup D_{xz}) = \{(x, y), \{(z, x)\}$. Let $W \in T^n$ satisfy $W \cap D_{xy} = (y^G, x^{-G})$. By A2 with $\xi = (x, y)$ and $\zeta = (z, x)$ applied to Y and W , $(z, x) \in f(Y)$ implies $(z, x) \in f(W)$. Hence, by Remark 2.3, for all $V \in T^n$, $(z, x) \in f(V)$. This contradicts A1. Case (ii). Let $Z \in L^n$ be such that $Z \cap D_{zy} = (z^G, y^{-G})$ and $f(Z) \cap D_{zy} = \{(y, z)\}$. By Remark 2.3, $Y \in L^n$ with $Y \cap D_{xy} = (x^G, y^{-G})$ and $Y \cap D_{xz} = (z^G, x^{-G})$ yields $f(Y) \cap (D_{xy} \cup D_{zy}) = \{(x, y), \{(y, z)\}$. If $W \in T^n$ satisfies $W \cap D_{xy} = (y^G, x^{-G})$, by A2 with $\xi = (x, y)$ and $\zeta = (y, z)$ applied to Y and W , $(y, z) \in f(W)$. Therefore, by Remark 2.3, for all $V \in T^n$, $(y, z) \in f(V)$. This contradicts A1. ■

Lemma 3.3. If $G \subset N$ is almost decisive for $x \in A$ against $y \in A \setminus \{x\}$ in f_L then G is decisive for x against y in f_L .

Proof. Suppose G is almost decisive for x against y . Select $J \subseteq \mathbb{M}G$. Let $Z \in L^n$ satisfy $Z \cap D_{xy} = (x^{G \cup J}, y^{-(G \cup J)})$. By Remark 2.3, it is enough to prove that $f(S) \cap D_{xy} = \{(x, y)\}$, for any $S \in L^n$ with $S \cap D_{xy} = Z \cap D_{xy}$. Choose $z \in A \setminus \{x, y\}$ and $Q \in L^n$ with $Q \cap D_{xy} = (x^G, y^{-G})$ and $Q \cap D_{zy} = (y^G, z^{-G})$. By Lemma 3.2, $f(Q) \cap (D_{xy} \cup D_{zy}) = \{(x, y), (y, z)\}$. Let $S \in L^n$ agree with Q in D_{zy} and with Z in D_{xy} . By A2 with $\xi = (x, y)$ and $\zeta = (y, z)$ applied to Q and S , $(x, y) \in f(S)$ and, by Lemma 3.1, $(y, x) \notin f(S)$. ■

Lemma 3.4. If $G \subset N$ is decisive in f_L and has at least two members then some non-empty $J \subset G$ is decisive in f_L .

Proof. Let G be decisive in f_L and $\{J, K\}$, with $J \neq \emptyset \neq K$, be a partition of G . Choose $x \in A$, $y \in A \setminus \{x\}$, $z \in A \setminus \{x, y\}$ and $R \in L^n$ such that $R \cap D_{xy} = (x^G, y^{-G})$, $R \cap D_{xz} = (x^J, z^{-J})$ and $R \cap D_{zy} = (z^K, y^{-K})$. As G is decisive in f_L , $f(R) \cap D_{xy} = \{(x, y)\}$. There are two cases by Lemma 3.1. Case 1: $f(R) \cap D_{zy} = \{(z, y)\}$. By Remark 2.3, K is almost decisive for z against y in f_L . Hence, by Lemmas 3.2 and 3.3, K is decisive in f_L . Case 2: $f(R) \cap D_{zy} = \{(y, z)\}$. There are two additional cases by Lemma 3.1. Case 2a: $f(R) \cap D_{xz} = \{(x, z)\}$. Then J is almost decisive for x against z in f_L and, by Lemmas 3.2 and 3.3, J is decisive in f_L . Case 2b: $f(R) \cap D_{xz} = \{(z, x)\}$. Let $Q \in L^n$ differ from R in that, for all $i \in J$, $Q_i \cap D_{zy} = \{(z, y)\}$. By A2 applied to R and Q with $\xi = (y, z)$ and $\zeta = (z, x)$, $(y, z) \in f(Q) \cap D_{zy}$, which contradicts the fact that G is decisive in f_L . ■

Lemma 3.5. If $\{i\} \subset N$ is decisive in f_L then $\{i\}$ is a strong dictator in f .

PROOF. Let $\{i\}$ be decisive in f_L , $R \in T^n$ and $\alpha = (x, y) \in A^2$, with $x \neq y$. To show that $f(R) \cap D_{xy} = R_i \cap D_{xy}$, let $G = \{j \in \mathbb{M}\{i\} : D_{xy} \subset R_j\}$ and $H = \{j \in \mathbb{M}\{i\} : \{\alpha'\} = D_{xy} \cap R_j\}$. Being $\{i\}$ decisive in f_L , there is nothing to prove if $G = \emptyset$, so let $G \neq \emptyset$. Case 1: $R_i \cap D_{xy} = \{\alpha\}$. Let $Q \in L^n$ satisfy $Q \cap D_{xy} = (x^H, y^{-H})$. As $\{i\}$ is decisive in f_L , by A2 with $\xi = \zeta = \alpha$, $\alpha \in f(R)$. If $\alpha' \in f(R)$ then let $S \in L^n$ differ from Q in that, for all $j \in G$, $S_j \cap D_{xy} = \{\alpha'\}$. By applying A2 with $\xi = \zeta = \alpha'$ to S and R , $\alpha' \in f(S)$, contradicting the decisiveness of $\{i\}$ in f_L . Case 2: $R_i \cap D_{xy} = \{\alpha'\}$. Analogous to case 1. Case 3: $R_i \cap D_{xy} = D_{xy}$. Let $Q, S \in T^n$ differ from R in that $Q_i \cap D_{xy} = \{\alpha\}$ and $S_i \cap D_{xy} = \{\alpha'\}$. By cases 1 and 2, $f(Q) \cap D_{xy} = \{\alpha\}$ and $f(S) \cap D_{xy} = \{\alpha'\}$. Therefore, by A2 applied to R and Q with $\xi = \zeta = \alpha$, $\alpha \in f(R)$ and, by A2 applied to R and S with $\xi = \zeta = \alpha'$, $\alpha' \in f(R)$. ■

Proposition 3.6. If $n \geq 2 < m$, $f: T^n \rightarrow C$ is strongly dictatorial if, and only if, f satisfies A1 and A2.

Proof. “ \Rightarrow ” If i is a strong dictator, f such that, for all $R \in T^n$, $f(R) = R_i$ satisfies both A1 and A2 trivially. “ \Leftarrow ” Let $n \geq 2 < m$ and $f: T^n \rightarrow C$ satisfy A1 and A2. By Lemma 3.5, it is enough to show that some $\{i\}$ is decisive in f_L . Choose non-empty $G \subset N$, $x \in A$, $y \in A \setminus \{x\}$ and $R \in L^n$ with $R \cap D_{xy} = (x^G, y^{-G})$. There are two cases by Lemma 3.1. Case 1: $f_L(R) \cap D_{xy} = \{(x, y)\}$. By Remark 2.3, G is almost decisive for x against y in f_L ; by Lemma 3.2, G is almost decisive in f_L ; by Lemma 3.3, G is decisive in f_L ; and, by Lemma 3.4, for some $i \in G$, $\{i\}$ is decisive in f_L . Case 2: $f_L(R) \cap D_{xy} = \{(y, x)\}$. It is similarly proved that, for some $i \in N \setminus G$, $\{i\}$ is decisive in f_L . ■

Remark 3.7. Proposition 3.6 is valid if f_L rather than f is required to yield complete binary relations, because the completeness assumption is only used in Lemma 3.1 and just with respect to f_L .

Call a function $f: T^n \rightarrow C$ that satisfies A1 resolute. By Proposition 3.6, a resolute $f: T^n \rightarrow C$ is dictatorial if, and only if, the independence condition A2 holds. This makes precise a sense in which independence and dictatorship amount to the same. Saari (1998, pp. 248-255) questions the appropriateness of IIA for social welfare functions: he argues that IIA aggregation rules do not treat transitivity as a valuable input and, consequently, in ignoring this information, there is no reason for expecting transitive outcomes. But, paradoxically, by Proposition 3.6, when independence condition A2 is imposed on a resolute aggregation rule that does not take transitivity into account, transitivity emerges as the outcome of the aggregation.

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