The majority rule with a chairman

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Abstract
The relative majority rule is the less demanding majority concept and probably defines the benchmark voting rule. This paper considers a natural variation on the relative majority rule: the one in which a given individual, the chairman, can break ties. Three types of axiomatic characterizations are suggested: one of the majority rule with a tie-breaking chairman, two of the relative majority rule and another one of the rules that are either the relative majority rule or the relative majority with a chairman.

Keywords: Social welfare function, relative majority rule, tie-breaking rule, axiomatic characterization, two alternatives.

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1. Introduction

The relative majority rule is the voting system typically adopted by legislatures to make ordinary decisions. It is prescribed in Robert's Rules of Order as the default voting system: with the exception of motions for which more than a majority is required, "When a quorum is present, a majority vote, that is a majority of the votes cast, ignoring blanks, is sufficient for the adoption of any motion that is in order" (Art. VIII, http://www.constitution.org/rror/rror-08.htm#46).

Axiomatic characterizations contribute to the justification and the understanding of the object being characterized, since an axiomatic characterization identifies properties that are exclusively satisfied by the characterized object. May (1952, p. 682) suggested the classic axiomatic characterization of the relative majority rule: with a given set of voters, the relative majority rule is the only neutral (or symmetric), anonymous and monotonic voting rule. Fishburn (1973, p. 58; 1983) offers another two axiomatizations in the same framework. Aşan and Sanver (2002, p. 411), Woeginger (2003, p. 91; 2005, p. 9) and Miroiu (2004, p. 362) characterize the relative majority rule when the set of voters is variable, whereas Xu and Zhong (2009) present a characterization when the set of voters is variable but their votes are held fixed.

Since legislatures are decision-making bodies, to be useful, a voting system must minimize indecision. One way to make the relative majority rule more resolute consists of defining one of the options as the status quo, which is preserved unless it is defeated by the other option. Consequently, it is as if blank votes counted on behalf of one of the alternatives over which a decision must be made. This violates the property of neutrality, according to which reversing the votes reverses the result.

This paper considers another strategy to reduce the number of cases in which the relative majority rule yields a tie. The strategy suggested violates the property of anonymity and consists of attributing a casting vote to one of the voters. This privilege exists in some legislatures. For instance, both the Speaker of the British House of Commons and the Vice President of the United States, when acting as President of the United States Senate, hold casting votes. As examples involving the judicature, both the President of the International Court of Justice and the President of the Spanish Constitutional Court hold a casting vote.

The main aim of the paper is to provide an axiomatic characterization of the relative majority in which some individual, the chairman, is given a casting vote. Proposition

3.5 presents one such characterization. Though it requires nine axioms, the only one not satisfied by the relative majority rule merely states that not all the cases in which the two alternatives receive the same support end in a tie. It is also shown that the other eight axioms characterize the rules than are either the relative majority rule or the relative majority rule with a chairman. Finally, it is also suggested a characterization of the relative majority rule in which the monotonicity axiom in May's (1952) characterization is not postulated.

2. Definitions and axioms

For some natural number $n \geq 2$, the members of the set $I = \{1, \ldots , n\}$ designate individuals. There are two alternatives: α and β. A preference over $\{\alpha, \beta\}$ is represented by a number from the set $\{-1, 0, 1\}$. If the number is 1, α is preferred to β; if -1, β is preferred to α; if 0, α is indifferent to β. A preference profile is a function $p : I \to \{-1, 0, 1\}$ assigning a preference over $\{\alpha, \beta\}$ to each member of I. The set P is the set of all preference profiles. For $p \in P$ and $i \in I$, p_i abbreviates $p(i)$. Alternatively, $p \in P$ can be interpreted as a profile of votes: $p_i = 1$ means that individual i votes for α; $p_i = -1$, that i votes for β; and $p_i = 0$, that i abstains.

Definition 2.1. A social welfare function is a mapping $f : P \to \{-1, 0, 1\}$.

A social welfare function takes as input the preferences over $\{\alpha, \beta\}$ of all the individuals and outputs a collective preference over $\{\alpha, \beta\}$. Specifically, for $p \in P$: (i) $f(p) = 1$ means that, according to f, the collective I prefers α to β; (ii) $f(p) = -1$, that I prefers β to α; and (iii) $f(p) = 0$, that I is indifferent between α and β. When the components of p are interpreted as votes rather than preferences, $f(p) = 1$ means that α is the winning alternative, $f(p) = -1$ means that β is the winning alternative and $f(p) = 0$ means that there is a tie between α and β (so it can be interpreted that the decision between α and β is postponed).

For $p \in P$ and $a \in \{-1, 0, 1\}$, $n_a(p)$ is the number of members of the set $\{i \in I: p_i = a\}$. A preference profile p is polarized if $n_1(p) = n_{-1}(p)$ and is non-polarized if $n_1(p) \neq n_{-1}(p)$.

Definition 2.2. The majority rule is the social welfare function $\mu : P \to \{-1, 0, 1\}$ such that, for all $p \in P$: (i) if $n_1(p) > n_{-1}(p)$, then $\mu(p) = 1$; (ii) if $n_1(p) < n_{-1}(p)$, then $\mu(p) = -1$; and (iii) if $n_1(p) = n_{-1}(p)$, then $\mu(p) = 0$.

Definition 2.3. The majority rule with a chairman $c \in I$ is the social welfare function μ_c: $P \to \{-1, 0, 1\}$ such that, for all $p \in P$: (i) if $\mu(p) = 0$, then $\mu_c(p) = p_c$; and (ii) if $\mu(p) \neq 0$, then $\mu_c(p) = \mu(p)$.

For $i \in I$ and $a \in \{-1, 0, 1\}$, a^i stands for the preference $p_i = a$. Similarly, for $J \subseteq I$, a^J represents the assignment of preference a to all the individuals in J. On the other hand, for $J \subseteq I$ and $p \in P$, p_J is the restriction of p to J. For instance, for $p \in P$, $(a^J, p_{I\setminus J})$ is the preference profile q such that, for all $i \in J$, $q_i = a$ and, for all $i \in J$, $q_i = p_i$. Expressions like $(a^i, b^{I\setminus\{i\}})$, $(p_J, q_{I\setminus J})$ and, for $i \in I\setminus J$, $(-1^J, 1^i, 0^{(I\setminus J)\setminus\{i\}})$ should be interpreted analogously.

DIC. *Non-dictatorship*. There is no $i \in I$ such that, for all $p \in P$, $f(p) = p_i$.

UNA. *Unanimity*. For all $a \in \{-1, 0, 1\}$, $f(a^I) = a$.

EXC. *Exclusion of an alternative without support*. For all $a \in \{-1, 1\}$ and $i \in I$, $f(a^i, 0^{I\setminus\{i\}}) \neq -a$.

DIC is the requirement that there is no strong dictator, that is, an individual whose preference always coincides with the collective preference. UNA states that a preference held by all the individuals in the collective must define the preference of the collective. EXC asserts that if no individual prefers alternative c to alternative d, then the collective preference cannot declare c preferred to d. DIC, UNA and EXC define properties that are difficult to question if the aim of the social welfare function is to generate a collective preference that synthesizes the preferences of all the individuals. The following three axioms can be viewed as expressing some form of anonymity.

For $i \in I$, $j \in I\setminus\{i\}$ and $p \in P$, $p^{i\leftrightarrow j}$ is the preference profile obtained from p by permuting the preferences of individuals i and j. Formally, $p^{i\leftrightarrow j}$ is the member q of P such that $q_i = p_j$, $q_j = p_i$ and, for all $k \in I\setminus\{i, j\}$, $q_k = p_k$.

LIM. *Limited variability*. For all $i \in I$, $j \in I\setminus\{i\}$ and $p \in P$, $f(p^{i\leftrightarrow j}) \in \{f(p), p_i, p_j\}$.

ANO$_P$. *Conditional anonymity for polarized profiles*. For all $i \in I$ and $j \in I\setminus\{i\}$, if there is some polarized $p \in P$ such that $f(p^{i\leftrightarrow j}) = f(p)$, then, for all polarized $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$.

ANO$_{NP}$. *Conditional anonymity for non-polarized profiles*. For all $i \in I$ and $j \in I\backslash\{i\}$, if there is some non-polarized $p \in P$ such that $f(p^{i\leftrightarrow j}) = f(p)$, then, for all non-polarized $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$.

May's (1952) anonymity is the requirement that $f(p^{i\leftrightarrow j}) = f(p)$ always. LIM weakens this requirement: if the permutation of the preferences of two individuals alters the collective preference, then the new preference must coincide with the preference of one of the two individuals. ANO$_P$ weakens anonymity in two directions: first, for $f(p^{i\leftrightarrow j}) = f(p)$ to be always the case, it must be that $f(p^{i\leftrightarrow j}) = f(p)$ holds at least once; and second, the preference profiles involved are polarized. ANO$_{NP}$ is like ANO$_P$ but with respect to non-polarized preference profiles.

NEU. *Limited neutrality*. For all $a \in \{-1, 1\}$, $i \in I$ and $j \in I\backslash\{i\}$, $f(a^i, -a^j, 0^{I\backslash\{i,j\}}) = -f(-a^i, a^j, 0^{I\backslash\{i,j\}})$.

May's (1952) neutrality asserts that, for all $p \in P$, $f(-p_1, \dots, -p_n) = -f(p)$. NEU limits this property to the case of minimally polarized preference profiles, namely, profiles in which two individuals have opposite preference and the rest of individuals are indifferent.

IND. *Weak independence*. For all $p \in P$, $q \in P$, $r \in P$ and non-empty $J \subset I$, if $f(0^J, p_{I\backslash J}) = f(0^J, q_{I\backslash J}) = 0$, then $f(r_J, p_{I\backslash J}) = f(r_J, q_{I\backslash J})$.

Suppose $f(p_J, p_{I\backslash J}) = f(p_J, q_{I\backslash J})$. This suggests that the resulting collective preference does not depend on whether the group $I\backslash J$ has preferences $p_{I\backslash J}$ or $q_{I\backslash J}$ when the group J has preferences p_J. A requirement of independence could be that this happening once must happen always: for any preferences r_J of the group J, $f(r_J, p_{I\backslash J}) = f(r_J, q_{I\backslash J})$. But neither μ nor μ_c satisfies this requirement. For instance, with $I = \{c, i, j\}$ and $f \in \{\mu, \mu_c\}$, $f(1^i, 1^c, 0^j) = f(1^i, 0^c, 0^j) = 1$, whereas $f(1^i, 1^c, -1^j) = 1 \neq 0 = f(1^i, 0^c, -1^j)$. IND restricts independence to the case in which $f(0^J, p_{I\backslash J}) = f(0^J, q_{I\backslash J}) = 0$.

POL. *Polarization does not imply indifference*. For some polarized $p \in P$, $f(p) \neq 0$.

Remark 2.4. The majority rule satisfies DIC, UNA, EXC, LIM, ANO$_P$, ANO$_{NP}$, NEU, IND (since $f(0^J, p_{I\backslash J}) = f(0^J, q_{I\backslash J}) = 0$ makes $p_{I\backslash J}$ and $q_{I\backslash J}$ cancellable, $\mu(r_J, p_{I\backslash J}) = \mu(r_J, q_{I\backslash J}) = \mu(r_J, 0^{I\backslash J})$), and the negation $\neg$POL of POL.

Remark 2.5. The majority rule with a chairman satisfies DIC, UNA, EXC, LIM, ANO_P, ANO_{NP}, NEU, IND, and POL.

It should not be difficult to verify that, for any $c \in I$, μ_c satisfies DIC, UNA, EXC, ANO_P, ANO_{NP}, NEU, and POL. As regards LIM, if p is non-polarized, then, for all $i \in I$ and $j \in I\backslash\{i\}$, $\mu_c(p^{i\leftrightarrow j}) = \mu_c(p)$. If p is polarized and $c \notin \{i, j\}$, then $\mu_c(p^{i\leftrightarrow j}) = \mu_c(p)$. And if p is polarized and $c \in \{i, j\}$, then $\mu_c(p^{i\leftrightarrow j}) = p_c \in \{\mu_c(p), p_i, p_j\}$.

With respect to IND, observe that $\mu_c(p) = 0$ is equivalent to $p = (0^I)$ or p polarized. Hence, $\mu_c(0^J, p_{I\backslash J}) = \mu_c(0^J, q_{I\backslash J}) = 0$ implies that i's preference is represented by 0. If $(r_J, 0^{I\backslash J})$ is polarized, then both $(r_J, p_{I\backslash J})$ and $(r_J, q_{I\backslash J})$ are polarized. As a result, if $c \in J$, $\mu_c(r_J, p_{I\backslash J}) = r_c = \mu_c(r_J, q_{I\backslash J})$. And if $c \notin J$, $\mu_c(r_J, p_{I\backslash J}) = p_c = 0 = q_c = \mu_c(r_J, q_{I\backslash J})$. If $(r_J, 0^{I\backslash J})$ is non-polarized, then both $(r_J, p_{I\backslash J})$ and $(r_J, q_{I\backslash J})$ are non-polarized. In view of this, $\mu_c(r_J, p_{I\backslash J}) = \mu(r_J, p_{I\backslash J})$ and $\mu_c(r_J, q_{I\backslash J}) = \mu(r_J, q_{I\backslash J})$. As $\mu_c(0^J, p_{I\backslash J}) = \mu_c(0^J, q_{I\backslash J}) = 0$ implies $\mu(0^J, p_{I\backslash J}) = \mu(0^J, q_{I\backslash J}) = 0$, it must be that $\mu(r_J, p_{I\backslash J}) = \mu(r_J, q_{I\backslash J})$.

Remark 2.6. DIC does not follow from UNA, EXC, LIM, ANO_P, ANO_{NP}, NEU, IND, and POL.

In fact, for $n \geq 3$, the strongly dictatorial social welfare function such that, for some $i \in I$ and all $p \in P$, $f(p) = p_i$ satisfies the eight axioms but not DIC.

3. Results

Given a social welfare function f, $i \in I$ and $j \in I\backslash\{i\}$, define the subset of individuals $\{i, j\}$ to be anonymous if, for all <u>polarized</u> $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$.

Lemma 3.1. If a social welfare function satisfies UNA, IND, NEU and POL, then, for some $i \in I$ and $j \in I\backslash\{i\}$, $\{i, j\}$ is not anonymous.

Proof. Suppose not:

$$\text{for all } i \in I \text{ and } j \in I\backslash\{i\},\ \{i, j\} \text{ is anonymous.} \tag{1}$$

By POL, let $p \in P$ be polarized and satisfy $f(p) \neq 0$. Define $I_1 = \{i \in I: p_i = 1\}$ and $I_{-1} = \{i \in I: p_i = -1\}$. Given that p is polarized, I_1 and I_{-1} have the same number t of elements. Choose $i \in I_1$ and $j \in L_{-1}$. By (1), $f(1^i, -1^j, 0^{I\setminus\{i, j\}}) = f(-1^i, 1^j, 0^{I\setminus\{i, j\}})$. By NEU, $f(1^i, -1^j, 0^{I\setminus\{i, j\}}) = -f(-1^i, 1^j, 0^{I\setminus\{i, j\}})$. Accordingly,

$$\text{for all } i \in I_1 \text{ and } j \in L_{-1},\ f(1^i, -1^j, 0^{I\setminus\{i, j\}}) = 0. \qquad (2)$$

This already contradicts $f(p) \neq 0$ if $t = 1$. If $t \geq 2$ then, taking (2) as the base case of an induction argument, choose $r \in \{2, \ldots, t\}$ and assume that, for all $s \in \{1, \ldots, r-1\}$, $J \subset I_1$ with s members and $K \subset L_{-1}$ with s members, $f(1^J, -1^K, 0^{I\setminus(J \cup K)}) = 0$. It must be shown that, for all $J \subseteq I_1$ with r members and $K \subseteq L_{-1}$ with r members, $f(1^J, -1^K, 0^{I\setminus(J \cup K)}) = 0$. To this end, let $J \subseteq I_1$ and $K \subseteq L_{-1}$ both have r members. Choose $i \in J$ and $j \in K$. By the induction hypothesis, $f(1^{J\setminus\{i\}}, -1^{K\setminus\{j\}}, 0^{I\setminus(J\setminus\{i\} \cup K\setminus\{j\})}) = 0$. By UNA, $f(0^I) = 0$. Hence, $f(1^{J\setminus\{i\}}, -1^{K\setminus\{j\}}, 0^{I\setminus(J\setminus\{i\} \cup K\setminus\{j\})}) = f(0^I)$. By IND, $f(1^J, -1^K, 0^{I\setminus(J \cup K)}) = f(1^i, -1^j, 0^{I\setminus\{i, j\}})$. By (2), $f(1^i, -1^j, 0^{I\setminus\{i, j\}}) = 0$. In sum, $f(1^J, -1^K, 0^{I\setminus(J \cup K)}) = 0$. The final conclusion is that $f(1^{I_1}, -1^{L_{-1}}, 0^{I\setminus(I_1 \cup L_{-1})}) = 0$. Since $p = (1^{I_1}, -1^{L_{-1}}, 0^{I\setminus(I_1 \cup L_{-1})})$, $f(p) \neq 0$ is contradicted.■

Lemma 3.2. Let social welfare function f satisfy LIM, NEU and ANO_P. Suppose $n \geq 3$. Then, for all $a \in \{-1, 1\}$, if $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) = a$, then: (i) $\{i, j\}$ is not anonymous; (ii) for all $k \in I\setminus\{i, j\}$, $\{i, k\}$ is not anonymous; and (iii) for all $k \in I\setminus\{i, j\}$, $\{j, k\}$ is anonymous.

Proof. Suppose $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) = a \neq 0$. (i) If $\{i, j\}$ is anonymous, then $f(-a^i, a^j, 0^{I\setminus\{i, j\}}) = a$. But, by NEU, $f(-a^i, a^j, 0^{I\setminus\{i, j\}}) = -f(a^i, -a^j, 0^{I\setminus\{i, j\}}) = -a$: contradiction. (ii) Suppose otherwise: $\{i, k\}$ is anonymous. Therefore, $f(a^i, -a^k, 0^{I\setminus\{i, k\}}) = f(-a^i, a^k, 0^{I\setminus\{i, k\}})$. By NEU, $f(a^i, -a^k, 0^{I\setminus\{i, k\}}) = -f(-a^i, a^k, 0^{I\setminus\{i, k\}})$. In consequence, $f(a^i, -a^k, 0^{I\setminus\{i, k\}}) = 0$. Given this, by LIM, $f(a^i, -a^j, 0^{I\setminus\{j, k\}}) \in \{f(a^i, -a^k, 0^{I\setminus\{i, k\}}), -a, 0\} = \{0, -a\}$. This contradicts the initial assumption that $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) = a$. (iii) Suppose not: $\{j, k\}$ is not anonymous. By (i) and (ii), neither $\{i, j\}$ nor $\{i, k\}$ is anonymous. Thus, by ANO_P, $\{i, k\}$ not anonymous implies $f(a^k, -a^j, 0^{I\setminus\{j, k\}}) \neq f(a^i, -a^j, 0^{I\setminus\{i, j\}}) = a$. By LIM, $f(a^k, -a^j, 0^{I\setminus\{j, k\}}) \in \{f(a^i, -a^j, 0^{I\setminus\{i, j\}}), 0, a\} = \{0, a\}$. As a result, $f(a^k, -a^j, 0^{I\setminus\{j, k\}}) = 0$. By NEU, $f(-a^k, a^j, 0^{I\setminus\{j, k\}}) = -f(a^k, -a^j, 0^{I\setminus\{j, k\}})$. Therefore, $f(-a^k, a^j, 0^{I\setminus\{j, k\}}) = 0$. By ANO_P, $f(a^k, -a^j, 0^{I\setminus\{j, k\}}) = f(-a^k, a^j, 0^{I\setminus\{j, k\}})$ implies that $\{j, k\}$ is anonymous: contradiction.■

Lemma 3.3. If a social welfare function f satisfies UNA, IND, LIM, NEU, ANO_P, and POL, then there is $i \in I$ such that, for all $p \in P$, $\mu(p) = 0$ implies $f(p) = p_i$.

Proof. By Lemma 3.1, for some $i \in I$ and $j \in I\setminus\{i\}$, $\{i, j\}$ is not anonymous. Hence, by ANO_P, $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) \neq f(-a^i, a^j, 0^{I\setminus\{i, j\}})$. By NEU, $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) = -f(-a^i, a^j, 0^{I\setminus\{i, j\}})$. In view of this, $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) \neq 0$. Consequently, $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) \in \{a, -a\}$. Without loss of generality, suppose $f(a^i, -a^j, 0^{I\setminus\{i, j\}}) = a$. By Lemma 3.2, for all $k \in I\setminus\{i\}$, $\{i, k\}$ is not anonymous and, for all $k \in I\backslash\{i, j\}$, $\{j, k\}$ is anonymous. Therefore, for all $k \in I\backslash\{i\}$, $f(a^i, -a^k, 0^{I\backslash\{i, k\}}) = a$. This and NEU imply that, for all $k \in I\backslash\{i\}$, $f(-a^i, a^k, 0^{I\backslash\{i, k\}}) = -a$. In sum,

$$\text{for all } k \in I\backslash\{i\} \text{ and } a \in \{-1, 1\}, f(a^i, -a^k, 0^{I\backslash\{i, k\}}) = a. \quad (3)$$

Let $p \in P$ satisfy $\mu(p) = 0$. This means that p is polarized or $p = (0^I)$. If $p = (0^I)$, then, by UNA, $f(p) = 0 = p_i$. If p is polarized, then define $I_1 = \{k \in I: p_k = 1\}$ and $I_{-1} = \{k \in I: p_k = -1\}$, where I_1 and I_{-1} have the same number $t \geq 1$ of members. Case 1: $p_i = 0$. Choose $r \in I_1$ and $s \in I_{-1}$. As $i \notin \{r, s\}$, $\{r, s\}$ is anonymous. Because of this, $f(1^r, -1^s, 0^{I\backslash\{r, s\}}) = f(-1^r, 1^s, 0^{I\backslash\{r, s\}})$. By NEU, $f(1^r, -1^s, 0^{I\backslash\{r, s\}}) = -f(-1^r, 1^s, 0^{I\backslash\{r, s\}})$. It then follows that,

$$\text{for all } r \in I_1 \text{ and } s \in I_{-1}, f(1^r, -1^s, 0^{I\backslash\{r, s\}}) = 0. \quad (4)$$

Taking (4) as the base case of an induction argument, choose $v \in \{2, \ldots, t\}$ and assume that, for all $w \in \{1, \ldots, v - 1\}$, $J \subset I_1$ with w members and $K \subset I_{-1}$ with w members, $f(1^J, -1^K, 0^{I\backslash(J \cup K)}) = 0$. It must be shown that, for all $J \subseteq I_1$ with v members and $K \subseteq I_{-1}$ with v members, $f(1^J, -1^K, 0^{I\backslash(J \cup K)}) = 0$. To this end, let $J \subseteq I_1$ and $K \subseteq I_{-1}$ both have v members. Choose $r \in J$ and $s \in K$. By the induction hypothesis, $f(1^{J\backslash\{r\}}, -1^{K\backslash\{s\}}, 0^{I\backslash(J\backslash\{r\} \cup K\backslash\{s\})}) = 0$. By UNA, $f(0^I) = 0$. By IND, $f(1^{J\backslash\{r\}}, -1^{K\backslash\{s\}}, 0^{I\backslash(J\backslash\{r\} \cup K\backslash\{s\})}) = f(0^I) = 0$ yields $f(1^J, -1^K, 0^{I\backslash(J \cup K)}) = f(1^r, -1^s, 0^{I\backslash\{r, s\}})$. By (4), $f(1^r, -1^s, 0^{I\backslash\{r, s\}}) = 0$, for which reason $f(1^J, -1^K, 0^{I\backslash(J \cup K)}) = 0$. All in all, this proves that $f(p) = 0 = p_i$. Case 2: $p_i = b \neq 0$. Since p is polarized, there is $r \in I$ such that $p_r = -b$. Let $q \in P$ differ from p only in that $p_i = p_r = 0$. By case 1, $f(q) = 0$. By UNA, $f(0^I) = 0$. By IND, it follows from $f(q) = f(0^I) = 0$ that $f(p) = f(b^i, -b^r, 0^{I\backslash\{i, r\}})$. By (3), $f(b^i, -b^r, 0^{I\backslash\{i, r\}}) = b$. As a result, $f(p) = p_i$.■

Lemma 3.4. For $n = 2$, if a social welfare function f satisfies DIC, UNA, EXC, IND, NEU, ANO_{NP}, and POL then, for some $c \in I$, $f = \mu_c$.

Proof. Let $I = \{i, j\}$. By UNA, for all $a \in \{-1, 0, 1\}$, $f(a^i, a^j) = a$. By POL, $f(1^i, -1^j) \neq 0$ or $f(-1^i, 1^j) \neq 0$. By NEU, $f(1^i, -1^j) = -f(-1^i, 1^j)$. Therefore, $f(1^i, -1^j) = 1$ or $f(1^i, -1^j) = -1$. Suppose $f(1^i, -1^j) = 1$ (the other possibility is handled analogously). By NEU, $f(-1^i, 1^j) = -1$. By EXC, for all $a \in \{-1, 1\}$, $f(a^i, 0^j) \in \{0, a\}$. If $f(a^i, 0^j) = 0$, then, by IND, $f(0^i, 0^j) = f(a^i, 0^j) = 0$ implies $f(0^i, -a^j) = f(a^i, -a^j)$. Since $f(a^i, -a^j) = a$, $f(0^i, -a^j) = a$ contradicts EXC. Consequently, for all $a \in \{-1, 1\}$, $f(a^i, 0^j) = a$.

Consider finally $f(0^i, a^j)$, where $a \in \{-1, 1\}$. By EXC, $f(0^i, a^j) \in \{0, a\}$. Suppose $f(0^i, a^j) = 0$. If $f(0^i, -a^j) = -a$, then $f(-a^i, 0^j) = -a$ and ANO_{NP} yield $f(a^i, 0^j) = f(0^i, a^j)$:

contradiction. Therefore, $f(0^i, -a^j) \neq -a$. By EXC, $f(0^i, -a^j) = 0$. In conclusion, for all $p \in P$, $f(p) = p_i$, which contradicts DIC. Accordingly, for all $a \in \{-1, 1\}$, $f(0^i, a^j) = a$.■

Proposition 3.5. A social welfare function f satisfies DIC, UNA, EXC, IND, LIM, NEU, ANO_P, ANO_NP, and POL if and only if, for some $c \in I$, $f = \mu_c$.

Proof. "$\Leftarrow$" Remark 2.5. "$\Rightarrow$" Lemma 3.4 covers the case $n = 2$, so let $n \geq 3$. By Lemma 3.3,

$$\text{there is } c \in I \text{ such that, for all } p \in P,\ \mu(p) = 0 \text{ implies } f(p) = p_c. \tag{5}$$

It then remains to be shown that, for all $p \in P$ such that $\mu(p) \neq 0$, $f(p) = \mu(p)$.

Part 1: for all $a \in \{-1, 1\}$, $f(a^c, 0^{I\setminus\{c\}}) = a$. Suppose not: for some $a \in \{-1, 1\}$, $f(a^c, 0^{I\setminus\{c\}}) \neq a$. By EXC, $f(a^c, 0^{I\setminus\{c\}}) \neq -a$. Therefore, $f(a^c, 0^{I\setminus\{c\}}) = 0$. By UNA, $f(0^I) = 0$. By IND, for any $i \in I\setminus\{c\}$, $f(a^c, 0^{I\setminus\{c\}}) = f(0^I) = 0$ implies $f(a^c, -a^i, 0^{I\setminus\{c, i\}}) = f(-a^i, 0^{I\setminus\{i\}})$. By (5), $f(a^c, -a^i, 0^{I\setminus\{c, i\}}) = a$. In sum, $f(-a^i, 0^{I\setminus\{i\}}) = a$, which contradicts EXC.

Part 2: for all $i \in I\setminus\{c\}$, $j \in I\setminus\{i, c\}$ and non-polarized $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$. Let $q \in P$ be polarized, with $q_c = 0$, $q_i = 1$ and $q_j = -1$. By (5), $f(q) = 0$. By UNA, $f(0^I) = 0$. By IND, $f(q) = f(0^I) = 0$ implies $f(1^c, q_{I\setminus\{c\}}) = f(1^c, 0^{I\setminus\{c\}})$. By Part 1, $f(1^c, 0^{I\setminus\{c\}}) = 1$. Therefore, $f(1^c, q_{I\setminus\{c\}}) = 1$. Let $r = q^{i\leftrightarrow j}$. By (5), $f(r) = 0$. By IND, $f(r^{i\leftrightarrow j}) = f(0^I) = 0$ implies $f(1^c, r_{I\setminus\{c\}}) = f(1^c, 0^{I\setminus\{c\}})$. Since, by Part 1, $f(1^c, 0^{I\setminus\{c\}}) = 1$, it follows that $f(1^c, r_{I\setminus\{c\}}) = 1 = f(1^c, q_{I\setminus\{c\}})$. As $(1^c, r_{I\setminus\{c\}})^{i\leftrightarrow j} = (1^c, q_{I\setminus\{c\}})$ and $(1^c, r_{I\setminus\{c\}})$ is non-polarized, by ANO_NP, for all non-polarized $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$.

Part 3: for all $i \in I\setminus\{c\}$ and non-polarized $p \in P$, $f(p^{i\leftrightarrow c}) = f(p)$. Suppose not: for some $i \in I\setminus\{c\}$ and non-polarized $p \in P$, $f(p^{i\leftrightarrow c}) \neq f(p)$. By ANO_NP, for all non-polarized $p \in P$, $f(p^{i\leftrightarrow c}) \neq f(p)$. In particular, for all $a \in \{-1, 1\}$, $f(a^c, 0^{I\setminus\{c\}}) \neq f(a^i, 0^{I\setminus\{i\}})$. Hence, by EXC and Part 1, for all $a \in \{-1, 1\}$, $f(a^i, 0^{I\setminus\{i\}}) = 0$. This and Part 2 imply that, for all $a \in \{-1, 1\}$ and $i \in I\setminus\{c\}$, $f(a^i, 0^{I\setminus\{i\}}) = 0$. The aim is to contradict DIC.

Step 1: for all $J \subseteq I\setminus\{c\}$ and $a \in \{-1, 1\}$, $f(a^J, 0^{I\setminus J}) = 0$. Let $a \in \{-1, 1\}$. As just shown, for all $i \in I\setminus\{c\}$, $f(a^i, 0^{I\setminus\{i\}}) = 0$. Taking this as the base case of an induction argument, choose cardinality t (not greater than the cardinality of $I\setminus\{c\}$) and assume that, for every non-empty $J \subset I\setminus\{c\}$ with $s < t$ members, $f(a^J, 0^{I\setminus J}) = 0$. Pick $J \subseteq I\setminus\{c\}$ with t elements. It must be shown that $f(a^J, 0^{I\setminus J}) = 0$. By UNA, $f(0^I) = 0$. Select $i \in J$. By the induction hypothesis, $f(a^{J\backslash\{i\}}, 0^{I\backslash(J\backslash\{i\})}) = 0$. By IND, $f(a^{J\backslash\{i\}}, 0^{I\backslash(J\backslash\{i\})}) = f(0^I) = 0$ implies $f(a^J, 0^{I\backslash J}) = f(a^i, 0^{I\backslash\{i\}})$. By the induction hypothesis, $f(a^i, 0^{I\backslash\{i\}}) = 0$. Consequently, $f(a^J, 0^{I\backslash J}) = 0$.

Step 2: for all $J \subseteq I\backslash\{c\}$, $K \subseteq I\backslash(J \cup \{c\})$ and $a \in \{-1, 1\}$, $f(a^J, -a^K, 0^{I\backslash(J \cup K)}) = 0$. By Step 1, $f(a^J, 0^{I\backslash J}) = f(-a^K, 0^{I\backslash K}) = 0$. By UNA, $f(0^I) = 0$. By IND, $f(a^J, 0^{I\backslash J}) = f(0^I) = 0$ implies $f(a^J, -a^K, 0^{I\backslash(J \cup K)}) = f(-a^K, 0^{I\backslash K})$. As $f(-a^K, 0^{I\backslash K}) = 0$, $f(a^J, -a^K, 0^{I\backslash(J \cup K)}) = 0$.

Step 3: for all $p \in P$, $f(p) = p_c$ (which contradicts DIC). Let $p \in P$. By Step 1 or Step 2, $f(0^c, p_{I\backslash\{c\}}) = 0$. This already proves that $f(p) = p_c$ if $p_c = 0$. If $p_c = a \neq 0$, then, by UNA, $f(0^I) = 0$. It then follows from IND and $f(0^c, p_{I\backslash\{c\}}) = f(0^I) = 0$ that $f(a^c, p_{I\backslash\{c\}}) = f(a^c, 0^{I\backslash\{c\}})$. By Part 1, $f(a^c, 0^{I\backslash\{c\}}) = a$. In view of this, $f(p) = f(a^c, p_{I\backslash\{c\}}) = a = p_c$.

Part 4: for all $p \in P$, $\mu(p) \neq 0$ implies $f(p) = \mu(p)$. Let $p \in P$ satisfy $\mu(p) = a \neq 0$. Case 1: for all $i \in I$, $p_i \neq -a$. If, in addition, for all $i \in I$, $p_i \neq 0$, then, by UNA, $f(p) = a$. If, for some $i \in I$, $p_i = 0$, then, by Part 1 and Part 3, for all $i \in I$, $f(a^i, 0^{I\backslash\{i\}}) = a$. Letting this be the base case of an induction argument, pick cardinality $t \leq n$ and assume that, for every non-empty $J \subset I$ with $s < t$ members, $f(a^J, 0^{I\backslash J}) = a$. Choose $J \subseteq I$ having t elements. It must be shown that $f(a^J, 0^{I\backslash J}) = a$. Suppose not: $f(a^J, 0^{I\backslash J}) \neq a$. By EXC, $f(a^J, 0^{I\backslash J}) = 0$. By Part 1 and Part 3, $n \neq 2$, so $n \geq 3$. By Part 3, it can be assumed that $c \in J$. By Part 1 and Part 3, $J \neq \{c\}$. Therefore, there is $j \in J\backslash\{c\}$. By UNA, $J \neq I$. Select then $i \in I\backslash J$. By UNA, $f(0^I) = 0$. By IND, $f(a^J, 0^{I\backslash J}) = f(0^I) = 0$ implies $f(a^J, -a^i, 0^{(I\backslash J)\backslash\{i\}}) = f(-a^i, 0^{I\backslash\{i\}})$. By Part 1 and Part 3, $f(-a^i, 0^{I\backslash\{i\}}) = -a$. To sum up, $f(a^J, -a^i, 0^{(I\backslash J)\backslash\{i\}}) = -a$.

By (5), $f(a^j, -a^i, 0^{I\backslash\{i, j\}}) = 0$. By UNA, $f(0^I) = 0$. By IND, $f(a^j, -a^i, 0^{I\backslash\{i, j\}}) = f(0^I) = 0$ implies $f(a^J, -a^i, 0^{(I\backslash J)\backslash\{i\}}) = f(a^{J\backslash\{j\}}, 0^{I\backslash(J\backslash\{j\})})$. By the induction hypothesis, $f(a^{J\backslash\{j\}}, 0^{I\backslash(J\backslash\{j\})}) = a$. In conclusion, $f(a^J, -a^i, 0^{(I\backslash J)\backslash\{i\}}) = a$: contradiction.

Case 2: for some $i \in I$, $p_i = -a$. Since $\mu(p) = a$, there are non-empty $J \subset I$ and $K \subseteq I\backslash J$ having the same number of members and some, possibly empty, $L \subset I\backslash(J \cup K)$ such that $p = (a^J, -a^K, 0^L, a^{I\backslash(J \cup K \cup L)})$. By Part 3, it can be assumed that $c \in I\backslash(J \cup K)$. By (5), $f(a^J, -a^K, 0^{I\backslash(J \cup K)}) = 0$. By UNA, $f(0^I) = 0$. By IND, $f(a^J, -a^K, 0^{I\backslash(J \cup K)}) = f(0^I) = 0$ implies $f(p) = f(0^{J \cup K \cup L}, a^{I\backslash(J \cup K \cup L)})$. By case 1, $f(0^{J \cup K \cup L}, a^{I\backslash(J \cup K \cup L)}) = a$. Hence, $f(p) = a$.■

The probably most salient feature of Proposition 3.5 is that too many axioms are invoked. This notwithstanding, it is worth pointing out that, since eight of the nine axioms in Proposition 3.5 are satisfied by the majority rule, the remaining axiom, the mild condition POL, is enough to direct the axioms at the majority rule with a chairman.

Of the three axioms in May's (1952) characterization, the majority rule with a chairman does not satisfy anonymity. The conjunction of ANO_{NP} and ANO_P is a possible candidate to replace anonymity. Yet, a strongly dictatorial social welfare function satisfies POL, ANO_{NP}, ANO_P, neutrality and monotonicity. And even if DIC is assumed, the resulting set of axioms does not characterize the majority rule with a chairman, as Example 3.6 proves.

Example 3.6. With $I = \{1, 2, 3\}$, let f be the social welfare function such that: (i) for every non-polarized p, $f(p) = \mu(p)$; (ii) $f(1^1, -1^2, 0^3) = 1$ and $f(-1^1, 1^2, 0^3) = -1$; (iii) $f(1^1, 0^2, -1^3) = -1$ and $f(-1^1, 0^2, 1^3) = 1$; and (iv) $f(0^1, -1^2, 1^3) = -1$ and $f(0^1, 1^2, -1^3) = 1$. It should not be difficult to verify that f is not the majority rule with a chairman and that f satisfies neutrality, monotonicity and all the axioms in Proposition 3.5 except LIM: $f(1^1, 0^2, -1^3) = -1$ but $f(1^1, -1^2, 0^3) = 1 \notin \{f(1^1, 0^2, -1^3), 0 , -1\}$.

Though the following result is probably not interesting by itself (because $\neg$POL is a strong and questionable requirement), it contributes to add value to Proposition 3.5: by Corollary 3.8, the eight axioms DIC, UNA, EXC, IND, LIM, NEU, ANO_P, and ANO_{NP} characterize the social welfare functions that are either the majority rule or the majority rule with a chairman, provided there are at least three individuals.

Proposition 3.7. With $n \geq 3$, a social welfare function f satisfies UNA, EXC, IND, ANO_{NP}, and $\neg$POL if and only if f is the majority rule.

Proof. "$\Leftarrow$" Remark 2.4. "$\Rightarrow$" Part 1: for all $i \in I$, $j \in I\backslash\{i\}$ and non-anonymous $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$. Choose $i \in I$ and $j \in I\backslash\{i\}$. By $\neg$POL, $f(1^i, -1^j, 0^{I\backslash\{i, j\}}) = 0$. By UNA, $f(0^I) = 0$. By IND, for any $k \in I\backslash\{i, j\}$, $f(1^i, -1^j, 1^k, 0^{I\backslash\{i, j, k\}}) = f(1^k, 0^{I\backslash\{k\}})$. On the other hand, by $\neg$POL, $f(-1^i, 1^j, 0^{I\backslash\{i, j\}}) = 0$. Hence, $f(0^I) = 0$ and IND imply $f(-1^i, 1^j, 1^k, 0^{I\backslash\{i, j, k\}}) = f(1^k, 0^{I\backslash\{k\}})$. Therefore, $f(1^i, -1^j, 1^k, 0^{I\backslash\{i, j, k\}}) = f(-1^i, 1^j, 1^k, 0^{I\backslash\{i, j, k\}})$. This proves that, for some non-anonymous $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$. By ANO_{NP}, for all non-anonymous $p \in P$, $f(p^{i\leftrightarrow j}) = f(p)$.

Part 2: for all $i \in I$ and $a \in \{-1, 1\}$, $f(a^i, 0^{I\backslash\{i\}}) = a$. Suppose not: for some $i \in I$ and $a \in \{-1, 1\}$, $f(a^i, 0^{I\backslash\{i\}}) \neq a$. By EXC, $f(a^i, 0^{I\backslash\{i\}}) = 0$. By Part 1, for all $j \in I$, $f(a^j, 0^{I\backslash\{j\}}) = 0$. Taking this to be the base case of an induction argument, choose cardinality $c \geq 2$ and assume that, for every $k \in \{1, \ldots , c - 1\}$ and every subset J of I having k members, $f(a^J, 0^{I\backslash J}) = 0$. Choose $j \in I\backslash J$. It must be shown that $f(a^J, a^j, 0^{I\backslash(J\backslash\{j\})}) = 0$. By UNA, $f(0^I) = 0$. By the induction hypothesis, $f(a^J, 0^{I\backslash J}) = 0$. By IND, $f(a^J, a^j, 0^{I\backslash(J\backslash\{j\})}) = f(a^j, 0^{I\backslash\{j\}})$. As previously shown, $f(a^j, 0^{I\setminus\{j\}}) = 0$, for which reason $f(a^J, a^j, 0^{I\setminus(J\setminus\{j\})}) = 0$. The final conclusion of the induction argument is that $f(a^I) = 0$, which contradicts UNA.

Part 3: for all non-empty $J \subset I$ and $a \in \{-1, 1\}$, $f(a^J, 0^{I\setminus J}) = a$. Taking Part 2 as the base case of an induction argument, choose $J \subset I$ and assume that, for all non-empty $K \subset J$ with fewer elements than J and all $a \in \{-1, 1\}$, $f(a^K, 0^{I\setminus K}) = a$. With $a \in \{-1, 1\}$, suppose $f(a^J, 0^{I\setminus J}) \neq a$. By EXC, $f(a^J, 0^{I\setminus J}) = 0$. By UNA, $f(0^I) = 0$. Let $i \in I\setminus J$. By IND, $f(a^J, -a^i, 0^{(I\setminus J)\setminus\{i\}}) = f(-a^i, 0^{I\setminus\{i\}}) = -a$, the last equality by Part 2. Choose $j \in J$. By ¬POL, $f(-a^i, a^j, 0^{I\setminus\{i, j\}}) = 0$. By UNA, $f(0^I) = 0$. By IND, $f(a^J, -a^i, 0^{(I\setminus J)\setminus\{i\}}) = f(a^{J\setminus\{j\}}, 0^{I\setminus(J\setminus\{j\})})$. By the induction hypothesis, $f(a^{J\setminus\{j\}}, 0^{I\setminus(J\setminus\{j\})}) = a$. Thus, $f(a^J, -a^i, 0^{(I\setminus J)\setminus\{i\}}) = a$: contradiction.

Part 4: for all $p \in P$, $f(p) = \mu(p)$. Case 1: p is polarized. By ¬POL, $f(p) = 0 = \mu(p)$. Case 2: p is non-polarized. Then $p = (a^J, -a^K, 0^L)$, where J, K and L are mutually disjoint sets and $I = J \cup K \cup L$. If $J = K = \emptyset$, by UNA, $f(p) = 0 = \mu(p)$. If $J = \emptyset \neq K$ or $J \neq \emptyset = K$, then, by Part 3, $f(p) = \mu(p)$. If $J \neq \emptyset \neq K$, then, since p is non-polarized, either J has more elements than K or vice versa. Suppose, without loss of generality, that J has more elements than K. Then there is a non-empty $J' \subset J$ such that K and $J\setminus J'$ have the same number of members. By ¬POL, $f(a^{J\setminus J'}, 0^{J'}, -a^K, 0^L) = 0$. By UNA, $f(a^I) = 0$. By IND, $f(a^{J\setminus J'}, a^{J'}, -a^K, 0^L) = f(a^{J'}, 0^{I\setminus J'})$. By Part 3, $f(a^{J'}, 0^{I\setminus J'}) = a$. Consequently, $\mu(p) = a = f(a^{J\setminus J'}, a^{J'}, -a^K, 0^L) = f(a^J, -a^K, 0^L) = f(p)$.■

Letting $I = \{1, 2\}$, consider the social welfare function f such that $f(1^i, 0^j) = f(0^i, -1^j) = 0$ and, otherwise, $f(p) = \mu(p)$. Whereas f satisfies UNA, EXC, IND, ANO_{NP}, and ¬POL (and also DIC, LIM, NEU, and ANO_{P}), it is not the majority rule. This example proves that Proposition 3.7 does not hold for $n = 2$.

Corollary 3.8. With $n \geq 3$, a social welfare function f satisfies DIC, UNA, EXC, IND, LIM, NEU, ANO_{P}, and ANO_{NP} if and only if, either $f = \mu$ or, for some $c \in I$, $f = \mu_c$.

Proof. "⇐" Remarks 2.4 and 2.5. "⇒" If POL holds, then, by Proposition 3.5, for some $c \in I$, $f = \mu_c$. If POL does not hold, then, by Proposition 3.7, $f = \mu$.■

When discussing May's monotonicity, Campbell and Kelly (2000, p. 699) remark that "This is a very strong condition – and it is not at all clear why it should be imposed". Proposition 3.7 is also useful to provide a characterization of the majority rule that assumes the familiar properties of neutrality and anonymity but not monotonicity.

CHO. *Choice*. For all $p \in P$ and $a \in \{-1, 1\}$, if, for all $i \in I$, $p_i \in \{0, a\}$, then $f(p) \in \{p_1, \ldots, p_n\}$.

CHO is the combination of UNA and EXC: if each individual is either indifferent or prefers the same alternative to the other, then the collective preference is chosen from among the individuals' preferences.

Proposition 3.9. A social welfare function *f* satisfies neutrality, anonymity, IND, and CHO if and only if *f* is the majority rule.

Proof. "$\Leftarrow$" Easily verified together with Remark 2.4. "$\Rightarrow$" CHO implies UNA. Anonymity implies EXC and ANO_{NP}. Neutrality and anonymity imply ¬POL. In view of all this, by Proposition 3.7, $f = \mu$.■

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