On the non-nonequivalence of weak and strict preference

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Abstract

Rechenauer (2008) claims that weak preference is a better starting point for preference relations than strict preferences. This contention is challenged. Definitional connections between weak and strict preference are suggested that lead to the opposite conclusion. This is not taken as a justification of the superiority of strict over weak preference as the primitive preference relation, but just as evidence that the given connections make a structure richer than the other. It is also make precise the sense in which weak and strict preference can be considered equivalent. The idea just consists of taking into account the other two implicit binary relations, indifference and non-comparability.

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1. Introduction

This paper challenges the analysis and conclusions put forward by Rechenauer (2008), who contends that the concept of weak preference is a better starting point for preference relations than the concept of strict preference. As hinted at by Rechenauer (2008, p. 386), for both approaches to be equivalent, "Given an appropriate definitional connection between these two sorts of preference relations, one should end up in the same place". He adopts definitional connections under which a complete and transitive weak preference R implies that the strict preference P is asymmetric and negatively transitive, but under which the converse is not true. He also questions two definitional connections making the converse true. But he fails to observe that criticisms analogous to those raised against the latter connections can be used against his own connections. In fact, the choice of definitional connections between R and P exhibits an asymmetry that explains the alleged non-equivalence of R and P (and the apparent superiority of R).

2. On Rechenauer's result

Rechenauer (2008, p. 386) takes for granted (1) and (2), where I represents indifference, R represents weak preference and P represents strict preference.

$$xIy \leftrightarrow (xRy \wedge yRx) \tag{1}$$

$$xRy \leftrightarrow (xPy \vee xIy) \tag{2}$$

This already reveals a first asymmetry: why is indifference defined in terms of R but not also in terms of P? The only justification of (1) that Rechenauer (2008, p. 386) offers is that it is needed. But when discussing Fishburn's (1970, p. 13) result that P asymmetric and negatively transitive implies R complete and transitive, Rechenauer (2008, p. 387) questions the use of (3), which Fishburn (1970, p. 12) considers a definition, not an assumption.

$$xIy \leftrightarrow (\neg xPy \wedge \neg yPx) \tag{3}$$

(3) is criticized on the grounds that $(\neg xPy \wedge \neg yPx) \rightarrow xIy$ is completeness of R in disguise. This is not entirely true, since (3) implies completeness in the presence of (1) and (2). In any case, (1) is open to a similar criticism: it could be deemed asymmetry of P in disguise, given (1) and (4). Condition (4) asserts that the disjunction in (1) is the exclusive disjunction: either xPy or xIy.

$$\neg(xIy \wedge xPy) \quad (4)$$

Remark 1. If (1), (2) and (4) hold, then P is asymmetric.

Suppose not: xPy and yPx. By (2), $xPy \vee xIy \to xRy$. Therefore, xPy implies xRy. By (2), $yPx \vee yIx \to yRx$. Consequently, yPx implies yRx. By (1), xRy and yRx imply xIy. As a result, $xIy \wedge xPy$, which contradicts (4).

$$xRy \to \neg yPx \quad (5)$$

$$\neg yPx \to xRy \quad (6)$$

Rechenauer (2008, p. 388) proves that, assuming (1), (2) and (5), if R is complete and transitive, then P is asymmetric and negatively transitive. He also shows that the converse is not true. This asymmetric result motivates his contention that R is a better primitive than P.

It is curious that attention is not drawn to the fact that (2) and (5) suffice to prove the asymmetry of P. When discussing the possibility of considering (6), whose inclusion would make the converse of his result true, he observes that (6) is logically equivalent to completeness of R (claim that requires the presence of (2)). So (6) is criticized on the grounds that it is again incompleteness of R in disguise. This is another instance of asymmetric reasoning: (6) is rejected because, together with (2), implies that R is complete; but (5) is accepted despite the fact that, together with (2), implies that P is asymmetric.

After proving that (2) and (6) imply completeness of R, Rechenauer (2008, p. 387) asserts: "This is bad. For, using (1) [here, the conjunction of (5) and (6)] or (3) [here (6)] as a definitional connection between R and P, our theorem really says that if P is asymmetric and negatively transitive <u>and</u> R is complete, then R is complete and transitive". This notwithstanding, he does not apply the same measuring stick to his own result: since (2) and (5) imply asymmetry of P, it appears that he should have stressed that he has shown (page 388, lines 1 and 2) that if R is complete and transitive <u>and</u> P is asymmetric, then P is asymmetric and negatively transitive. So, by symmetry, (6) is challengeable if and only if (5) is.

What in any case should be pointed out is that a justifiable definitional connection between R and P need not be discarded just because it carries with it a property that one would like to obtain from other assumptions. Otherwise, following Remark 2, one would be forced to remove at least one of the three conditions (1), (2) and (4), all of which seem unobjectionable when one takes R as primitive.

Remark 2. If (1), (2) and (4) hold, then P is asymmetric.

Suppose not: xPy and yPx. By (2), xPy implies xRy. By (2), yPx implies yRx. By (1), xIy. By (4), $\neg xPy$: contradiction.

3. The opposite of Rechenauer's result

In Rechenauer's view, the conjunction of (2) and (6) produces an undesirable result. His response is to discard (6). An alternative approach consists of dispensing with (2). Remark 3 next suggests that (4) may be considered a good candidate: given (1) and (4), (6) cannot be deemed completeness in disguise (so Rechenauer's criticism against (6) could be viewed as excessive).

Remark 3. (1), (4) and (6) do not imply that R is complete.

Let $X = \{x, y, z\}$. The binary relation R on X is such that, for all $a \in X$ and $b \in X \backslash \{a\}$, $\neg aRb$. It is clear that R is not complete. The binary relation P on X is such that, for all $a \in X$ and $b \in X \backslash \{a\}$, aPb. (6) holds because, for all $a \in X$ and $b \in X \backslash \{a\}$, $\neg aRb$ and bPa. And (4) holds because, according to (1), aIb is never the case.

Proposition 4. Assume (1), (4) and (6). (i) If P is asymmetric and negatively transitive, then R is complete and transitive. (ii) The converse does not hold.

Proof. (i) Suppose $\neg yRx$. By (6), $\neg yRx \rightarrow xPy$. Hence, xPy. By asymmetry of P, $\neg yPx$. By (6), xRy. This proves the completeness of R: $\neg yRx$ implies xRy. Suppose R is not transitive: xRy, yRz and $\neg xRz$. By (6), $\neg xRz$ implies zPx. By asymmetry, $\neg xPz$. By (6), zRx. Since P is negatively transitive, zPx implies zPy or yPx. If yPx, then, by asymmetry, $\neg xPy$. By (6), yRx. Therefore, by (1), yRx and xRy imply yIx. But then having yIx and yPx contradicts (4). If zPy, then, by asymmetry, $\neg yPz$. By (6), zRy. Hence, by (1), zRy and yRz imply zIy. But then having zIy and zPy contradicts (4).

(ii) Let $X = \{x, y, z\}$. The complete and transitive binary relation R on X is such that xRy, yRz and xRz. The binary relation P on X is such that xPy, yPz, xPz and zPx. As xPz and

zPx, P is not asymmetric. In addition, $\neg zPy$, $\neg yPx$ and zPx, so P is not negatively transitive. To show that (6) holds, consider the two cases in which $\neg aPb$. With respect to $\neg zPy$, R is such that yRz; and, concerning, $\neg yPx$, R is such that xRy. On the other hand, consider the three cases in which $\neg aRb$. As regards $\neg yRx$, xPy holds; with respect to $\neg zRy$, yPz holds; and concerning $\neg zRx$, xPz holds. In sum, (6) is satisfied. Finally, (4) holds because, according to (1), aIb is never the case.■

Mimicking Rechenauer's strategy, one could take Proposition 4 as a reason to think of P as the better starting point for preference relations. But this conclusion, as well as Rechenauer's, is precipitate because it is the result of definitional connections between P and R that are, more or less obviously, biased against one of the two preference relations. It will argued in the next section that there are good reasons to accept the equivalence of weak and strict preference.

Before proceeding with the justification of the equivalence, another result that apparently points to the superiority of P over R is suggested. It relies on (7), obtained by inserting (3) into (2). As might be expected, (7) also implies (6): if $\neg yPx$, then ($xPy \vee \neg yPx$) holds and, by (7), xRy.

$$xRy \leftrightarrow (xPy \vee \neg yPx) \tag{7}$$

Proposition 5. Assume (7). (i) If R is complete and transitive, then P is asymmetric and negatively transitive. (ii) The converse does not hold.

Proof. (i) Part 1: if (7) holds, then R is complete. Suppose not: $\neg xRy \wedge \neg yRx$. By (7), $\neg xRy$ implies $\neg xPy \wedge yPx$. Similarly, by (7), $\neg yRx$ implies $\neg yPx \wedge xPy$. It then follows from ($\neg xRy \wedge \neg yRx$) that ($\neg xPy \wedge yPx \wedge \neg yPx \wedge xPy$), which leads to the contradiction $yPx \wedge \neg yPx$.

Part 2: if (7) holds and P is asymmetric and negatively transitive, then R is transitive. Assume xRy, yRz and $\neg xRz$. By (7),

$$\neg xPz \wedge zPx. \tag{8}$$

By (7), xRy implies $xPy \vee \neg yPx$. By (7), yRz implies $yPz \vee \neg zPy$. This leads to four possibilities. Case 1: $xPy \wedge yPz$. By asymmetry of P, xPy implies $\neg yPx$. By asymmetry of P, yPz implies $\neg zPy$. As P is negatively transitive, $\neg zPy$ and $\neg yPx$ imply $\neg zPx$, contradicting (8). Case 2: $xPy \wedge \neg zPy$. By (8) and negative transitivity, $\neg xPz$ and $\neg zPy$

imply $\neg xPy$, which contradicts xPy. Case 3: $\neg yPx \wedge yPz$. By (8) and negative transitivity, $\neg yPx$ and $\neg xPz$ imply $\neg yPz$, which contradicts yPz. Case 4: $\neg yPx \wedge \neg zPy$. By negative transitivity, $\neg zPx$, which contradicts (8).

(ii) Let $X = \{x, y, z\}$. The complete and transitive binary relation R on X is such that, for all $a \in X$ and $b \in X \backslash \{a\}$, aRb. The binary relation P on X is such that, for all $a \in X$ and $b \in X \backslash \{a\}$, aPb. Evidently, P is not asymmetric. Moreover, for all $a \in X$ and $b \in X \backslash \{a\}$, $aRb \leftrightarrow aPb$. In view of this, (7) holds.■

4. An equivalence result

Consider the definitional connections displayed in the following two tables, where N is the non-comparability relation.

	yPx	$\neg yPx$
xPy	xNy	xRy
$\neg xPy$	yRx	xIy

	yRx	$\neg yRx$
xRy	xIy	xPy
$\neg xRy$	yPx	xNy

The above may be regarded as natural definitional connections between the four relations involved. For instance, taking P as primitive, having both xPy and yPx expresses the incomparability between x and y, whereas having both $\neg xPy$ and $\neg yPx$ expresses the indifference between x and y. When R is taken as primitive, incomparability is represented by $\neg xRy$ and $\neg yRx$, whereas indifference is associated with xPy and yPx.

When one has been accustomed to thinking in terms of R, the definition of xIy as $\neg xPy \wedge \neg yPx$ may appear questionable. But, otherwise, what does $\neg xPy \wedge \neg yPx$ express? If it is argued that it may be interpreted as incomparability, then what does $xPy \wedge yPx$ represent? Incomparability as well? In this case, why not consider $xRy \wedge yRx$ also as expressing incomparability? If one accepts the view that the four combinations between xRy, $\neg xRy$, yRx and $\neg yRx$ represent different preference situations, then that view should also be accepted with respect to the four combinations between xPy, $\neg xPy$, yPx and $\neg yPx$. As one of those combinations must express indifference, the natural property of asymmetry prevents considering the case $xPy \wedge yPx$ as indifference.

$$xNy \leftrightarrow (xPy \wedge yPx) \tag{9}$$

$$xPy \leftrightarrow (\neg yRx \wedge \neg xNy) \tag{10}$$

(9) and (10) adapt (1) and (2) to the strict preference framework, with N playing the role of I.

Remark 6. If P is asymmetric, then (9) and (10) imply (5) and (6).

If P is asymmetric, then it is always true that $\neg xNy$. Hence, $(\neg yRx \wedge \neg xNy)$ is equivalent to $\neg yRx$. By (10), $\neg yRx$ equivalent to $(\neg yRx \wedge \neg xNy)$ implies that $\neg yRx$ is equivalent to xPy. That is, $xPy \leftrightarrow \neg yRx$.

Proposition 7. If (9) and (10) hold, then: (i) P is asymmetric if and only if R is complete; and (ii) P is negatively transitive if and only if R is transitive.

Proof. (i) Suppose P asymmetric. If $\neg xRy$, then, by Remark 6, yPx. By asymmetry, $\neg xPy$. By Remark 6, $\neg xPy$ implies yRx. Suppose R complete. If xPy and yPx, then, by (10), $xPy \rightarrow (\neg yRx \wedge \neg xNy)$ and $yPx \rightarrow (\neg xRy \wedge \neg yNx)$. Accordingly, xPy implies $\neg yRx$ and yPx implies $\neg xRy$, which contradicts completeness.

(ii) Let P be negatively transitive. Assume xRy and yRz. By (10), $xRy \vee yNx \rightarrow \neg yPx$. Hence, xRy implies $\neg yPx$. By (10), $yRz \vee zNy \rightarrow \neg zPy$. Therefore, yRz implies $\neg zPy$. As P is negatively transitive, $\neg zPy$ and $\neg yPx$ imply $\neg zPx$. By (10), $\neg zPx \rightarrow (xRz \vee zNx)$. By (9), $\neg zPx$ implies $\neg zNx$. As a result, xRz holds. This proves the transitivity of R. On the other hand, let R be transitive. Assume $\neg xPy$ and $\neg yPz$. By (10), $\neg xPy \rightarrow (yRx \vee xNy)$. Given $\neg xPy$, by (9), xNy does not hold. Consequently, $\neg xPy$ implies yRx. By (10), $\neg yPz \rightarrow (zRy \vee yNz)$. By (9), $\neg yPz$ yields $\neg xNy$, for which reason $\neg yPz$ implies zRy. In sum, zRy and yRx. By transitivity of R, zRx. By (10), $zRx \vee xNz \rightarrow \neg xPz$. In view of this, zRx implies $\neg xPz$, which shows P to be negatively transitive.■

Proposition 7 expresses the equivalence of weak and strict preference under (9) and (10) that those taking P. (9) states that having x strictly preferred to y and vice versa represents incomparability. (10) asserts that having x strictly preferred to y is equivalent to the conjunction of two conditions: that x and y are comparable and that y is not weakly preferred to x. The equivalence is strong in the sense that completeness of R is made equivalent to asymmetry of P, whereas transitivity of R is made equivalent to negative transitivity of P.

5. Another equivalence result

One may dislike Proposition 7 because of the atypical binary relation *N*. This problem can be solved by just inserting (9) into (10). Be that as it may, another equivalence result will be presented that relies on conditions involving only *P* and *R*.

$$xPy \rightarrow xRy \tag{11}$$

$$(xRy \wedge yRx) \leftrightarrow (\neg xPy \wedge \neg yPx) \tag{12}$$

It is difficult to raise objections against (11): strict preference implies weak preference. (12) is the outcome of identifying the two ways in which indifference is conceptualized in terms of strict and weak preference. (12) is the conjunction of (13) and (14).

$$(\neg xPy \wedge \neg yPx) \rightarrow (xRy \wedge yRx) \tag{13}$$

$$(xRy \wedge yRx) \rightarrow (\neg xPy \wedge \neg yPx) \tag{14}$$

Lemma 8. If (11) and (13) hold, then *R* is complete.

Proof. Suppose not: $\neg xRy$ and $\neg yRx$. By (11), $\neg xRy$ implies $\neg xPy$. By (11), $\neg yRx$ implies $\neg yPx$. Therefore, $\neg xPy \wedge \neg yPx$. By (13), $xRy \wedge yRx$: contradiction.■

Lemma 9. If (11) and (14) hold, then *P* is asymmetric.

Proof. Suppose not: xPy and yPx. By (11), xPy implies xRy. By (11), yPx implies yRx. Consequently, $xRy \wedge yRx$. By (14), $\neg xPy \wedge \neg yPx$: contradiction.■

Lemma 10. If (11) and (12) hold, then *P* negatively transitive implies *R* transitive.

Proof. Suppose not: xRy, yRz and $\neg xRz$. By (11), $\neg xRz$ implies $\neg xPz$. If $\neg zPx$, then, by (13), $xRz \wedge zRx$, which contradicts $\neg xRz$. Therefore, zPx. If yRx, then the assumption xRy and (14) imply $\neg xPy$ and $\neg yPx$. On the other hand, if $\neg yRx$, then, by (11), $\neg yPx$. Consequently, no matter whether yRx or $\neg yRx$, $\neg yPx$ holds. If $\neg zPy$, then $\neg yPx$ and the negative transitivity of *P* imply $\neg zPx$, contradicting the previous conclusion that zPx. Accordingly, zPy. By (11), zRy. By assumption, yRz. In view of this, by (14), $\neg zPy$ and $\neg yPz$, contradicting zPy.■

Lemma 11. If (11) and (12) hold, then *R* transitive implies *P* negatively transitive.

Proof. Suppose not: $\neg xPy$, $\neg yPz$ and xPz. By (11), xPz implies xRz. If zRx, then, by (14), $\neg xPz \wedge \neg zPx$, which contradicts xPz. Therefore, $\neg zRx$. If $\neg zPy$, then the assumption $\neg yPz$ and (13) imply zRy and yRz. On the other hand, if zPy, then, by (11), zRy. Consequently, zRy holds regardless of whether $\neg zPy$ or zPy. If $\neg yPx$, then the assumption $\neg xPy$ and (13) imply xRy and yRx. If, alternatively, yPx, then, by (11), yRx. Accordingly, yRx holds irrespectively of whether $\neg yPx$ or yPx. By transitivity of R, zRy and yRx imply zRx, which contradicts the previous conclusion that $\neg zRx$.■

Proposition 12. If (11) and (12) hold, then R is complete and transitive if and only if P is asymmetric and negatively transitive.

Proof. Lemmas 8, 9, 10 and 11.■

Remark 13. (12) cannot be weakened into (13) in Proposition 12.

Let $X = \{x, y, z\}$. The binary relation R on X is such that, for all $a \in X$ and $b \in X \backslash \{a\}$, aRb. The binary relation P on X is such that xPz. It is not difficult to verify that (11) and (13) hold. Despite this, P is not negatively transitive.

Remark 14. (12) cannot be weakened into (14) in Proposition 12.

With $X = \{x, y, z\}$, the binary relation R on X is such that xRy, yRz and xRz. The binary relation P on X is such that xPz. It is not difficult to verify that (11) and (14) hold. This notwithstanding, P is not negatively transitive.

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