# A research output index that values productivity

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**Abstract**
In a few years, the Hirsch index has become of the most famous research output indices. It is defined as the maximum number *h* of papers each of which has at least *h* citations. Hence, in an output with *n* papers and Hirsch index *h*, $n - h$ papers are irrelevant. This paper suggests and characterizes another index in which each paper counts. The index is motivated by the, perhaps counterintuitive, idea that additional citations of less cited papers should count more than additional citations of more cited papers. This idea can be justified by the principle of diminishing marginal productivity when the papers with fewer citations tend to be the more recent papers.

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## 1. Introduction

The Hirsch (2005) index has become one of the most popular bibliometric measures of an individual's research output. Define a research output $x$ to consist of a set of papers and, for each paper, the number of citations the paper has received. The Hirsch index, or $h$-index, of $x$ is the maximum number $h$ such that there are $h$ papers in $x$ each having at least $h$ citations.

There are several axiomatic characterizations of the Hirsch index; see Woeginger (2008a, 2008b) or Quesada (2009, 2010a, 2010b). There are also many indices defined from, or inspired by, the Hirsch index; see, for instance, Alonso et al. (2009). A bibliography on the Hirsch index and its variants can be found in Cabrerizo (2010). There are as well many reasons to criticize the Hirsch index. This paper focuses on one: that $n - h$ papers in an output with $n$ papers and Hirsch index $h$ are irrelevant.

For instance, it does not seem reasonable to consider the output consisting of one paper with one citation as equivalent to the output consisting of one hundred papers in which one paper is cited many times and each of the other ninety-nine just once. A consequence of this feature of the Hirsch index is that it may be considered too coarse, as it fails to discriminate between substantially different outputs. In addition, since it is each time harder to increase the Hirsch index, it appears to value too much what is exceptional (many papers cited many times) and underestimate what is the norm (most papers receive a few citations, if any).

Implicit in the Hirsch index is the general idea that the marginal value of a citation is eventually decreasing: speaking very loosely, a new citation is potentially more productive in a more cited paper than in any of the least cited papers (with the Hirsch index itself establishing the divide between more cited and less cited papers). For example, consider the output $x = (4, 3, 2, 1, 1)$ in which the most cited paper has 4 citations, the second most cited receive 3 and so on. The Hirsch index of $x$ is $h(x) = 2$. If a citation is added to the most cited paper, the index remains at 2. The same occurs if added to the second most cited. But adding a new citation to the third most cited paper raises the index to 3. The addition of a citation to any of the last two papers does not modify the index. As a result, the vector representing the impact on the index of a marginal citation is (0, 0, 1, 0, 0). This suggests that $h(x) = 2$ means that the value of the marginal citation can only be positive for paper $h(x) + 1$.

The index suggested in this paper is based on the idea of attributing a positive value to each marginal citation. What is more: this value should be increasing when papers are listed from most to least cited. In particular, the marginal impact vector (at least when a new citation does not alter the existing ranking among papers) is required to be, for the five-paper case, (1, 2, 3, 4, 5). This says that another citation of the most cited paper raises the index by 1, whereas another citation of the fifth most cited paper raises the index by 5.

To value more a citation of a less cited paper can be justified by the principle that if something is harder to obtain, its marginal value has to be higher. In the case at hand, being the least cited paper suggests that it is more difficult to that paper to obtain a citation, so each such citation should contribute more to the index. On the other hand, being the most cited paper suggests that it has been easier for this paper to receive a citation and, accordingly, a new citation should be less valuable.

A probably more appealing justification can be suggested when the more recent papers are those with a fewer number of citations. When this is the case, an additional citation of the last paper may be considered more valuable than a citation of any of the preceding papers because, by the principle of decreasing marginal productivity, it seems more difficult to produce another paper deserving a citation when one has already produced many such papers than when one has produced a few or none.

The suggested index is just the sum of the citations each one of them weighted by the position that the paper occupies in the citation ranking. For instance, the index of $x = (4, 3, 2, 1, 1)$ is $1{\cdot}4 + 2{\cdot}3 + 3{\cdot}2 + 4{\cdot}1 + 5{\cdot}1$. This index is characterized by four axioms. The first just declares the unit of measurement to be the index of the output consisting of one paper with one citation. The second axiom is additivity: the index of the sum of two outputs (with the same underlying ranking among papers) is the sum of the respective indices. The Hirsch index does not satisfy additivity but subadditivity: the index of the sum of two output is never greater than the sum of the indices of the outputs.

The third axiom captures the above evaluation principle: the impact on the index of the marginal citation of a more cited paper is smaller than the impact of the marginal citation of a less cited paper. The last axiom requires the index to take the minimum value consistent with the preceding three axioms. Quesada (2010b) shows that the Hirsch index (and also the Egghe (2006a, 2006b) index, a variant of the Hirsch index) can be characterized following the same approach: the index is obtained by minimizing with respect to a family of indices satisfying certain monotonicity properties.

## 2. Definitions and axioms

Let ℕ be the set of non-negative integers. Define $X$ to be the set of all vectors $x = (x_1, \dots , x_n)$ such that $n \in \mathbb{N}\backslash\{0\}$, $x_1 \geq x_2 \geq \dots \geq x_n$ and, for all $i \in \{1, \dots , n\}$, $x_i \in \mathbb{N}$. A member $x = (x_1, \dots , x_n)$ of $X$ represents a research output consisting of $n \geq 1$ papers and, for each paper $i$, the number $x_i$ of citations that $i$ receives. For $x \in X$, $d_x$ designates the number of components (or dimension) of vector $x$.

**Definition 2.1.** A research output index (or index, for short) is a mapping $f : X \to \mathbb{N}$.

**Definition 2.2.** The $h$-index (or Hirsch index) is the index $h$ such that, for all $x \in X$: (i) $x_1 = 0$ implies $h(x) = 0$; and (ii) $x_1 \neq 0$ implies $h(x) = max\{n \in \{1, \dots , d_x\}: x_n \geq n\}$.

**Definition 2.3.** The $q$-index is the index such that, for all $x \in X$, $q(x) = \sum_{i=1}^{d_x} i{\cdot}x_i$.

The $q$-index values every item in a research output: no paper and no citation are worthless. Its counterintuitive feature is perhaps weighting more the citation of a less cited paper than that of a more cited paper. A plausible justification of this feature can be offered when the ranking of the papers coincides with the other in which they are written. In this context, the $q$-index tries to capture the idea that what is more difficult to obtain should be more valuable.

Specifically, suppose that the research output consists of four cited papers. Consider the value of writing a fifth paper. That paper has been preceded by four other papers, which have explored territory no longer available to the fifth paper. It appears reasonable, in general, to expect that it will be more difficult to explore a new territory, that is, to produce a fifth paper that has to be differentiated from the previous four. In addition, the fifth paper must compete with the other four for citations. Thus, it seems more difficult for the fifth paper to get a citation that may go the other papers than for the first paper to receive its first citation when that citation does not have the possibility of going to non-existing papers.

The $q$-index is not proposed as an impact index but rather as a productivity index. In this respect, it tries to put a premium on normality (producing many papers with few citations), not excellence (producing papers with a large number of citations).

UNI. *Unit of measurement.* $f(1) = 1$.

UNI sets the unit of measurement: a paper with one citation is one unit of the index. For $x \in X$ and $y \in X$, define $x + y$ and $y + x$ to represent the member $z$ of $X$ such that: (i) for all $i \in \{1, \dots, min\{d_x, d_y\}\}$, $z_i = x_i + y_i$; and (i) for all $i \in \{min\{d_x, d_y\} + 1, \dots, max\{d_x, d_y\}\}$, $z_i = x_i$ if $d_x > d_y$ and $z_i = y_i$ if $d_x < d_y$.

ADD. *Additivity*. For all $x \in X$ and $y \in X$, $f(x + y) = f(x) + f(y)$.

Additivity expresses the view that the evaluation of a research output is separable (it also implies that the evaluation of a research output is subject to constant scale returns): no value of two outputs is lost by merging them.

For all $x \in X$ and $i \in \{1, \dots, d_x\}$, $(\dots, x_i + 1, \dots)$ represents the vector obtained from $x$ by replacing $x_i$ with $x_i + 1$. Similarly, $(\dots, x_i + 1, x_{i+1} + 1, \dots)$ represents the vector obtained from $x$ by replacing $x_i$ with $x_i + 1$ and $x_{i+1}$ with $x_{i+1} + 1$.

MON. *Monotonicity*. For all $x \in X$ and $i \in \{1, \dots, d_x - 1\}$ such that $(\dots, x_i + 1, \dots) \in X$, $f(\dots, x_i + 1, \dots) - f(x) < f(\dots, x_i + 1, x_{i+1} + 1, \dots) - f(\dots, x_i + 1, \dots)$.

MON is a particular case of the property that the impact of a marginal citation of paper $i$ is smaller than the impact of a marginal citation of paper $i + 1$, namely, that $(\dots, x_i + 1, \dots) \in X$ and $(\dots, y_{i+1} + 1, \dots) \in X$ imply $f(\dots, x_i + 1, \dots) - f(x) < f(\dots, y_{i+1} + 1, \dots) - f(y)$. Illustrating MON with a simple example, $f(1, 0) - f(0, 0) < f(1, 1) - f(1, 0)$: the first citation of the first paper is less valuable than the first citation of the second paper.

MIN. For each $x \in X$ and each index $f^*$ satisfying UNI, ADD and MON, $f(x) \leq f^*(x)$.

ADD and MON are expansive requirements: they establish conditions under which the index grows. MIN is a restrictive axiom: it sets a limit to the potential increase of the index. In particular, MIN demands the index to always adopt the smallest value consistent with the three previous axioms.

## 3. Result

**Remark 3.1.** The $q$-index satisfies UNI, ADD and MON.

Since $q(1) = 1 \cdot 1 = 1$, $q$ satisfies UNI. With respect to ADD, assume, without loss of generality, that $d_x \leq d_y$, so $q(x + y) = \sum_{i=1}^{d_x} i \cdot (x_i + y_i) + \sum_{i=d_x+1}^{d_y} i \cdot y_i = \sum_{i=1}^{d_x} i \cdot x_i + \sum_{i=1}^{d_y}$

$i{\cdot}y_i = q(x) + q(y)$. As regards MON, $(\ldots, x_i + 1, \ldots) \in X$ implies $q(\ldots, x_i + 1, \ldots) - q(x) = i(x_i + 1) + \sum_{j\neq i} j{\cdot}x_j - i{\cdot}x_i - \sum_{j\neq i} j{\cdot}x_j = i$. On the other hand, $q(\ldots, x_i + 1, x_{i+1} + 1, \ldots) - q(\ldots, x_i + 1, \ldots) = i(x_i + 1) + (i + 1)(x_{i+1} + 1) + \sum_{j\notin\{i,\, i+1\}} j{\cdot}x_j - i(x_i + 1) - (i + 1)x_{i+1} - \sum_{j\notin\{i,\, i+1\}} j{\cdot}x_j = i + 1$. As a result, $q(\ldots, x_i + 1, \ldots) - q(x) = i < i + 1 = q(\ldots, x_i + 1, x_{i+1} + 1, \ldots) - q(\ldots, x_i + 1, \ldots)$, which proves MON.

For $n \in \mathbb{N}\setminus\{0\}$, define $(0^{\leq n})$ to be the member of $X$ consisting of $n$ papers each one of which has no citation and $(1^{\leq n})$ to be the member of $X$ consisting of $n$ papers each one of which has exactly one citation. For $n \in \mathbb{N}\setminus\{0, 1\}$, $(1^{\leq n-2}, 0^{n-1}, 0^{n})$ is the member of $X$ consisting of $n$ papers in which $n - 2$ papers have one citation each and the last two have no citation. Similarly, $(1^{\leq n-1}, 0^{n})$ is the member of $X$ consisting of $n$ papers in which $n - 1$ papers have one citation each and the last one has no citation.

**Lemma 3.2.** If $f$ is an index that satisfies UNI, ADD and MON, then: (i) for all $n \in \mathbb{N}\setminus\{0\}$, $f(0^{\leq n}) = q(0^{\leq n})$ and $f(1^{\leq n}) \geq q(1^{\leq n})$; and (ii) for all $x \in X$, $f(x) \geq q(x)$.

*Proof*. Part 1: for all $n \in \mathbb{N}\setminus\{0\}$, $f(0^{\leq n}) = q(0^{\leq n})$. By ADD, $f(0^{\leq n} + 0^{\leq n}) = f(0^{\leq n}) + f(0^{\leq n}) = 2f(0^{\leq n})$. Since $f(0^{\leq n} + 0^{\leq n}) = f(0^{\leq n})$, $f(0^{\leq n}) = 2f(0^{\leq n})$. Consequently, $f(0^{\leq n}) = 0 = q(0^{\leq n})$.

Part 2: for all $n \in \mathbb{N}\setminus\{0\}$, $f(1^{\leq n}) \geq q(1^{\leq n})$. If $n = 1$, then, by UNI, $f(1) = 1 = q(1)$. Taking this result as the base case of an induction argument, choose $r \in \mathbb{N}\setminus\{0, 1\}$ and suppose that, for all $t \in \{1, \ldots, r - 1\}$, $f(1^{\leq t}) = q(1^{\leq t})$. It must be shown that $f(1^{\leq r}) = q(1^{\leq r})$. By MON, $f(1^{\leq r-1}, 0^{r}) - f(1^{\leq r-2}, 0^{r-1}, 0^{r}) < f(1^{\leq r}) - f(1^{\leq r-1}, 0^{r})$, where $(1^{\leq r-2}, 0^{r-1}, 0^{r}) = (0, 0)$ if $r = 2$. Hence, $f(1^{\leq r}) > 2f(1^{\leq r-1}, 0^{r}) - f(1^{\leq r-2}, 0^{r-1}, 0^{r})$. By ADD, $f(1^{\leq r-1}, 0^{r}) = f(1^{\leq r-1}) + f(0^{\leq r})$. By the induction hypothesis, $f(1^{\leq r-1}) = q(1^{\leq r-1})$. By Part 1, $f(0^{\leq r}) = 0$. In conclusion, $f(1^{\leq r-1}, 0^{r}) = q(1^{\leq r-1})$.

Case 2a: $r = 2$. In this case, $f(1^{\leq r}) = f(1, 1)$, $f(1^{\leq r-1}, 0^{r}) = f(1, 0)$ and $f(1^{\leq r-2}, 0^{r-1}, 0^{r}) = f(0, 0)$. By case 1, $f(0, 0) = 0$. Accordingly, $f(1, 1) > 2q(1, 0) = 2$. In sum, $f(1, 1) \geq 3 = q(1, 1)$. Case 2b: $r > 2$. By ADD, $f(1^{\leq r-2}, 0^{r-1}, 0^{r}) = f(1^{\leq r-2}) + f(0^{\leq r})$. By the induction hypothesis, $f(1^{\leq r-2}) = q(1^{\leq r-2})$. By Part 1, $f(0^{\leq r}) = 0$. Accordingly, $f(1^{\leq r-2}, 0^{r-1}, 0^{r}) = q(1^{\leq r-2})$. It then follows from $f(1^{\leq r}) > 2f(1^{\leq r-1}, 0^{r}) - f(1^{\leq r-2}, 0^{r-1}, 0^{r})$ that $f(1^{\leq r}) > 2q(1^{\leq r-1}) - q(1^{\leq r-2}) = 2\sum_{i=1}^{r-1} i - \sum_{i=1}^{r-2} i = \sum_{i=1}^{r-1} i + (r - 1) = \sum_{i=1}^{r} i - 1 = q(1^{\leq r}) - 1$. Since $f(1^{\leq r}) \in \mathbb{N}$, $f(1^{\leq r}) > q(1^{\leq r}) - 1$ implies $f(1^{\leq r}) \geq q(1^{\leq r})$.

Part 3: for all $x \in X$, $f(x) \geq q(x)$. Part 1 deals with the case $x_1 = 0$. Taking this as the base case of an induction argument, choose $n \in \mathbb{N}\setminus\{0\}$ and suppose that, for all $r \in \{0, \ldots, n - 1\}$, $x_1 = r$ implies $f(x) \geq q(x)$. It must be shown that, for all $x \in X$ such that $x_1 = n$, $f(x)$

$\geq q(x)$. To this end, let $x \in X$ satisfy $x_1 = n$. Define $i \in \{1, \dots, d_x\}$ to be such that $x_i = n$ and either $i = d_x$ or $x_{i+1} < n$ ($i$ is the largest number of papers in $x$ having the maximum number $n$ of citations). Let $y \in X$ be obtained by removing one citation to every paper in the set $\{1, \dots, i\}$: for all $j \in \{1, \dots, i\}$, $y_j = x_i - 1$ and, for all $j \in \{i + 1, \dots, d_x\}$, $y_j = x_i$. By ADD, $f(x) = f(y) + f(1^{\leq i})$. By the induction hypothesis, $f(y) \geq q(y)$. By Part 2, $f(1^{\leq i}) \geq q(1^{\leq i})$. As a consequence, $f(x) \geq q(y) + q(1^{\leq i})$. By Remark 3.1, $q$ satisfies ADD. In view of this, $y + (1^{\leq i}) = x$ implies $q(y) + q(1^{\leq i}) = q(x)$. Summarizing, $f(x) \geq q(x)$.■

**Proposition 3.3.** An index $f$ satisfies UNI, ADD, MON and MIN if and only if $f = q$.

*Proof*. "⇐" By Remark 3.1, $q$ satisfies UNI, ADD and MON. This and Lemma 3.2(ii) imply that $q$ satisfies MIN. "⇒" By Lemma 3.2(ii), for all $x \in X$, $f(x) \geq q(x)$. Since $q$ satisfies UNI, ADD and MON, by MIN, $f(x) = q(x)$.■

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