Consistency, unanimity and dictatorship

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Abstract

Social choice functions, social choice correspondences and social welfare functions are shown to possess a hierarchy of dictators when preferences are strict and axioms of unanimity, consistency with respect to the removal of alternatives and consistency with respect to the removal of individuals hold. When preferences are non-strict, the same axioms imply, in each case, the existence of a hierarchy of weak dictators. These results can be viewed as signs of the tension between consistency and the dispersion of decision power. They also provide a unifying theme for Arrow's theorem and the Gibbard-Satterthwaite theorem.

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1. Introduction

Arrow's (1963, p. 97) theorem and the Gibbard (1973) - Satterthwaite (1975) theorem are two fundamental results in economic theory. Arrow's theorem is a result on social welfare functions, which are rules that aggregate preferences. The theorem establishes conditions under which an individual has the power to transfer his strict preference to the aggregate preference. The Gibbard-Satterthwaite theorem is a result on social choice functions, which are rules that determine a collective decision from preferences over the possible decisions. The theorem establishes conditions under which an individual has the power to determine the collective decision.

It is know that the two theorems are closely related results. For instance, a single proof is suggested by Reny (2001), whereas a general theorem of which the two results are special cases is presented in Barberà (2001). This paper provides an additional connection based on the concept of consistent social choice.

Consistency principles pervade many decision-making fields. In cooperative game theory, consistency is a natural property of solutions; see Thomson (1990). In non-cooperative game theory, it is an axiom to characterize equilibrium concepts and the foundation of a solution for extensive form games (subgame perfection); see Peleg and Tijs (1996) and Selten (1975). In axiomatic bargaining theory, it takes the form of the independence of irrelevant alternatives condition that characterizes the Nash (1950) bargaining solution. In the theory of the allocation of indivisible goods, consistency contributes to generate hierarchical allocation structures; see Ergin (2000). And, in social choice theory, the independence of irrelevant alternatives condition in Arrow's theorem itself can also be viewed as a requirement of consistency defining the aggregation of preferences over *n* alternatives in terms of the aggregation over just two.

Broadly defined, consistency means that the solution of a certain problem is related to the solution of subproblems of the given problem. Since a social choice problem involves individuals and alternatives, subproblems of the general problem can be obtained by removing alternatives or by removing individuals. When alternatives are removed, a social choice can be said to be consistent if the removal of a discarded alternative does not alter the selected option. This is essentially what independence of irrelevant alternatives type conditions demand. When individuals are removed, it is easier to justify a consistency requirement when there are only two alternatives. Thus, a social choice over just two alternatives can also be said to be consistent if the removal of an individual not favouring the selected option does not modify the choice made.

It is shown that these two consistency conditions, together with a requirement of unanimity, make social choice functions hierarchically dictatorial: there is a ranking of the individuals such that, for any social choice problem involving any subset I of individuals, the member of I appearing higher in the ranking determines the social choice. When preferences are not strict, the hierarchy of dictators becomes a hierarchy of weak dictators, which means that the member of I appearing higher in the ranking has veto power when the social choice is determined.

The same results hold for social choice correspondences and social welfare functions when the consistency requirement with respect to alternatives is conveniently adapted; see Section 4. For social choice correspondences, consistency demands that the alternatives chosen when alternative x has been removed is obtained from the set of chosen alternatives by removing x. For social welfare functions, consistency requires that the aggregation of a preference restriction coincide with the restriction of the aggregation of the unrestricted preferences. For the social choice correspondence that can be naturally defined from a social welfare function, this consistency property (which is closely related to the independence of irrelevant alternatives condition in Arrow's theorem) implies the consistency property for social choice correspondences.

The results in Section 4 suggest that the same principles lead to the same conclusions in three social choice frameworks that differ in the level of complexity of the outcome. Social choice functions represent the lowest level of complexity, since the preference information associated with the individuals must be transformed into just an alternative. The next level corresponds to social choice correspondences, which are asked to generate a set of alternatives. Finally, social welfare functions define the higher complexity level, as they are required to produce a preference. The common principles are unanimity and two axioms of consistency, one with respect to individuals and the other with respect to alternatives. The common conclusion is that a hierarchy of dictators emerges when preferences are strict, and a hierarchy of weak dictators emerges when indifference is allowed.

Arrow's theorem and the Gibbard-Satterthwaite theorem could both be viewed as particular instances of these results, because the two theorems can be stated in terms of unanimity plus a second condition implied by the consistency axioms postulated here: in Arrow's theorem, the condition is independence of irrelevant alternatives; in the Gibbard-Satterthwaite theorem, it is strategy proofness. So both theorems exemplify the relationship between consistency and dictatorship in the presence of unanimity.

2. Definitions

The $n \geq 2$ members of the finite set N represent individuals. A society is a non-empty subset of N having at least two members. The $m \geq 3$ members of the finite set A represent alternatives. A choice set is a non-empty subset of A. A preference p on choice set C is a sequence $(C_1, \dots, C_r)$ of choices sets such that: (i) $C_1 \cup \dots \cup C_r = C$; and (ii) for all $s \in \{1, \dots, r\}$ and $t \in \{1, \dots, r\}\backslash\{s\}$, $C_s \cap C_t = \varnothing$. The interpretation is that, for $s \leq r$, each alternative in C_s is indifferent to the rest of alternatives in C_s and, for $s < r$, is preferred to each alternative in $C_{s+1} \cup \dots \cup C_r$. For preference $p = (C_1, \dots, C_r)$ on C and $k \in \{1, \dots, r\}$, ${}^kp$ designates the kth element in the sequence p. Hence, ${}^1p$ is the set of most preferred alternatives in p, ${}^2p$ is the set of the second most preferred alternatives in p, and so on. A preference $p = (C_1, \dots, C_r)$ on C is strict if, for all $s \in \{1, \dots, r\}$, C_s has only one element.

For choice set C, let T_C be the set of preferences that can be defined on C. A preference profile on C for society I is a function $P_I^C : I \rightarrow T_C$ assigning a preference on C to each member of I. Let T be the set of all preference profiles P_I^C such that I is a society and C a choice set. For $P_I^C \in T$ and $i \in I$, P_i^C will abbreviate $P_I^C(i)$. A preference profile P_I^C such that, for all $i \in I$, P_i^C is a strict preference is called a strict preference profile. Let L be the set of all strict preference profiles P_I^C such that I is a society and C a choice set. For preference $p = (C_1, \dots, C_r)$ on C, the restriction $p|_D$ of p to choice set $D \subseteq C$ is the sequence obtained from $(C_1 \cap D, \dots, C_r \cap D)$ by removing all the empty components. The restriction P_J^D of $P_I^C \in T$ to society $J \subseteq I$ and choice set $D \subseteq C$ is the preference profile $Q_J^D : J \rightarrow T_D$ such that, for all $i \in J$, $Q_J^D(i) = P_I^C(i)|_D$. For choice set $D \subseteq C$, $P_I^C|_D$ also designates the restriction of $P_I^C \in T$ to D.

Definition 2.1. Letting A^* be set of all non-empty subsets of A, a social choice rule on $X \subseteq T$ is a mapping $f : X \rightarrow A^*$ such that, for all $P_I^C \in X$, $f(P_I^C) \subseteq C$. A social choice function on $X \subseteq T$ is a social choice rule such that, for all $P_I^C \in X$, $f(P_I^C)$ has only one member.

Definition 2.2. A social choice rule f on $X \subseteq T$ has a hierarchy of dictators if there is a sequence $(i_1, \dots, i_n)$ listing the n members of N without repetitions such that, for all $P_I^C \in X$, $f(P_I^C) = {}^1P_{i_k}^C$, where $k = min\{r \in \{1, \dots, n\}: i_r \in I\}$.

Definition 2.3. A social choice rule f on $X \subseteq T$ has a hierarchy of weak dictators if there is a sequence $(i_1, \dots, i_n)$ listing the n members of N without repetitions such that, for all $P_I^C \in X$, $f(P_I^C) \subseteq {}^1P_{i_k}^C$, where $k = min\{r \in \{1, \dots, n\}: i_r \in I\}$.

A preference profile P_I^C represents a social choice problem in which the members of society I must select a set of alternatives from C when, for all $i \in I$, P_i^C represents the preference of individual i on choice set C. A social choice rule f on X associates with every member P_I^C of X a subset $f(P_I^C)$ of C representing the choice set selected by the members of I.

If f has a hierarchy of dictators $(i_1, \dots, i_n)$, the set of selected alternatives in social choice problem P_I^C is given by the set of most preferred alternatives of the member of I listed first in the hierarchy. When the dictators are weak, the set of selected alternatives may be a strict subset of the set of most preferred alternatives of that individual.

Definition 2.4. A social welfare function on $X \subseteq T$ is a mapping $F : X \to T$ such that, for all $P_I^C \in X$, $F(P_I^C) \in T_C$.

Definition 2.5. For social welfare function F on $X \subseteq T$, ${}^1F$ is the social choice rule $f : X \to A^*$ such that, for all $P_I^C \in X$, $f(P_I^C) = {}^1F(P_I^C)$.

A social welfare function F on X associates with every member P_I^C of X a preference $F(P_I^C)$ representing the preference of society I on C. The social choice rule ${}^1F$ is obtained from social welfare function F by associating with P_I^C the set of most preferred alternatives in $F(P_I^C)$.

Definition 2.6. A social welfare function F on $X \subseteq T$ has a hierarchy of dictators if there is a sequence $(i_1, \dots, i_n)$ listing the n members of N without repetitions such that, for all $P_I^C \in X$, $F(P_I^C) = P_{i_k}^C$, where $k = min\{r \in \{1, \dots, n\}: i_r \in I\}$.

Definition 2.7. A social welfare function F on $X \subseteq T$ has a hierarchy of weak dictators if there is a sequence $(i_1, \dots, i_n)$ listing the n members of N without repetitions such that, for all $P_I^C \in X$, $x \in C$ and $y \in C\backslash\{x\}$, if $k = min\{r \in \{1, \dots, n\}: i_r \in I\}$ and x is preferred to y in $P_{i_k}^C$, then x is preferred to y in $F(P_I^C)$.

When F has a hierarchy of dictators, $F(P_I^C)$ coincides with the member of I higher in the hierarchy. Having a hierarchy of weak dictators means that x is preferred to y in $F(P_I^C)$ if the member of I higher in the hierarchy that is not indifferent between x and y prefers x to y. Hence, dictators impose the full preference, whereas weak dictators just impose strict preference.

3. Axioms

UNA. *Unanimity*
For all $P_I^C \in T$, if there is $x \in C$ such that, for all $i \in I$, ${}^1P_i^C = \{x\}$, then $f(P_I^C) = \{x\}$.

UNA states that if all the members of the society regard some alternative as the most preferred, then that alternative should constitute the social choice. UNA captures the idea that the social choice should be minimally respectful with the individuals' preferences. In that sense, it could be viewed as a basic requirement of consistency between preferences and choices.

CAL. *Consistency with respect to alternatives*
For all $P_I^C \in T$ and $x \in C$, $f(P_I^C)\backslash\{x\} \neq \varnothing$ implies $f(P_I^{C\backslash\{x\}}) = f(P_I^C)\backslash\{x\}$.

CAL is the consistency condition according to which the solution to social choice problem $P_I^{C\backslash\{x\}}$ is the restriction of the solution of problem P_I^C to choice set $C\backslash\{x\}$, provided the restriction is non-empty. This is a strong form of consistency. A weaker consistency requirement could be that $x \notin f(P_I^C)$ implies $f(P_I^{C\backslash\{x\}}) = f(P_I^C)$, which asserts that a discarded alternative is irrelevant. When f is a social choice function, the two consistency properties are equivalent.

CIN. *Basic consistency with respect to individuals*
For all $P_I^C \in L$ such that C has two elements and $i \in I$, if ${}^1P_i^C \neq f(P_I^C)$ and $f(P_I^C)$ is a singleton, then $f(P_{I\backslash\{i\}}^C) = f(P_I^C)$.

It is in general not obvious when an individual may be considered irrelevant, in the sense that the removal of the individual should not alter the social choice. CIN identifies irrelevant individuals for the simple social choice problem in which there are only two alternatives, only one of them can be chosen and preferences are strict. In this case, CIN states that any individual not favouring the social choice can be discarded: if $\{x\}$ is the social choice with the opposition of individual i, then $\{x\}$ must remain the social choice when i's opinion is not taken into account.

MON. *Monotonicity for choice sets with two alternatives*
For all $P_I^C \in L$ such that C has two elements and $i \in I$, if ${}^1P_i^C \neq f(P_I^C)$ and ${}^1Q_i^C = f(P_I^C)$, then $f(Q_i^C, P_{I\setminus\{i\}}^C) = f(P_I^C)$.

Consider again social choice problems in which preferences are strict, there are two alternatives and only one of the alternatives can be chosen. MON states that more support to the winning alternative cannot lead to a rejection of that alternative.

Remark 3.1. MON implies CIN.

Suppose $P_I^C \in L$ and $i \in I$ are such that $C = \{x, y\}$ and $\{x\} = {}^1P_i^C \neq f(P_I^C) = \{y\}$. If $f(P_{I\setminus\{i\}}^C) \neq f(P_I^C)$, then $f(P_{I\setminus\{i\}}^C) = \{x\}$. By MON, $f(P_{I\setminus\{i\}}^C) = \{x\}$ implies $f(P_I^C) = \{x\}$: contradiction.

Definition 3.2. A social choice function f on $X \subseteq T$ is strategy-proof if, for all $P_I^C \in X$, $i \in I$ and $Q_i^C \in X$, the member of $f(Q_i^C, P_{I\setminus\{i\}}^C)$ is not preferred to the member of $f(P_I^C)$ in P_i^C, where $(Q_i^C, P_{I\setminus\{i\}}^C)$ is the preference profile obtained from P_I^C by replacing P_i^C with Q_i^C.

Remark 3.3. A social choice function f on L that satisfies CAL and either CIN or MON is strategy-proof.

With $f(P_I^C) = \{y\}$, suppose $i \in I$ prefers $x \in C$ to y in P_i^C. To show that, for all $Q_i^C \in T$, $f(Q_i^C, P_{I\setminus\{i\}}^C) \neq \{x\}$, let $Q_i^C \in T$ and assume $f(Q_i^C, P_{I\setminus\{i\}}^C) = \{x\}$. With $D = \{x, y\}$, by CAL, $f(P_I^C) = \{y\}$ implies $f(P_I^D) = \{y\}$, whereas $f(Q_i^C, P_{I\setminus\{i\}}^C) = \{x\}$ implies $f(Q_i^D, P_{I\setminus\{i\}}^D) = \{y\}$. If i prefers x to y in Q_i^C, then $Q_i^C = P_i^C$, so it must be that $f(Q_i^D, P_{I\setminus\{i\}}^D) = f(P_I^D)$: contradiction. If i prefers y to x in Q_i^C and CIN holds, then, by CIN, $f(Q_i^D, P_{I\setminus\{i\}}^D) = \{x\}$ implies $f(P_{I\setminus\{i\}}^D) = \{x\}$, whereas $f(P_I^D) = \{y\}$ implies $f(P_{I\setminus\{i\}}^D) = \{y\}$: contradiction. Lastly, if i prefers y to x in Q_i^C and MON holds, then, by MON, $f(P_I^D) = \{y\}$ implies $f(Q_i^D, P_{I\setminus\{i\}}^D) = \{y\}$: contradiction.

UNA′. *Unanimity*
For all $P_I^C \in T$, if there is $x \in C$ such that, for all $i \in I$, ${}^1P_i^C = \{x\}$, then ${}^1F(P_I^C) = \{x\}$.

CIN′. *Basic consistency with respect to individuals*

For all $P_I^C \in L$ such that C has two elements and $i \in I$, if $P_i^C \neq F(P_I^C) \in L$, then $F(P_{I\setminus\{i\}}^C) = F(P_I^C)$.

CAL′. *Consistency with respect to alternatives*
For all $P_I^C \in T$ and $x \in C$, if $C\setminus\{x\} \neq \emptyset$, then $F(P_I^{C\setminus\{x\}}) = F(P_I^C)|_{C\setminus\{x\}}$.

UNA′, like UNA, asserts that x should be the socially most preferred alternative when every individual considers x the most preferred alternative. CIN′, like CIN, restricts attention to the case in which there are only two alternatives and all the involved preferences are strict. For this case, CIN′ holds that the social preference is not altered by removing an individual whose preference completely disagrees with the social preference. CAL′ introduces a more substantial change in CAL by letting the social preference associated with $P_I^{C\setminus\{x\}}$ coincide with the restriction to $C\setminus\{x\}$ of the social preference that corresponds to P_I^C. Hence, the solution of the restricted aggregation problem $P_I^{C\setminus\{x\}}$ is the solution of the aggregation problem P_I^C restricted to $C\setminus\{x\}$.

Definition 3.4. A social welfare function F on $X \subseteq T$ is independent of irrelevant alternatives if, for all $P_I^C \in X$, $Q_I^C \in X$, $x \in C$, and $y \in C\setminus\{x\}$, $P_I^C|_{\{x,y\}} = Q_I^C|_{\{x,y\}}$ implies $F(P_I^C)|_{\{x,y\}} = F(Q_I^C)|_{\{x,y\}}$.

Remark 3.5. A social welfare function F that satisfies CAL′ is independent of irrelevant alternatives.

With $D = \{x, y\}$, suppose $P_I^C|_D = Q_I^C|_D$. Hence, $F(P_I^D) = F(Q_I^D)$. By CAL′, $F(P_I^D) = F(P_I^C)|_D$ and $F(Q_I^D) = F(Q_I^C)|_D$. Therefore, $F(P_I^C)|_D \neq F(Q_I^C)|_D$ would imply $F(P_I^D) \neq F(Q_I^D)$: contradiction.

CAL″. For all $P_I^C \in T$ and non-empty $B \subset C$, ${}^1F(P_I^B) = {}^1(F(P_I^C)|_B)$.

Since the step between CAL′ and independence of irrelevant alternatives seems too short, it may be interesting to consider other consistency properties not so obviously connected with independence. CAL″ is one possible such property, asserting that, for $B \subset C$, the most preferred alternatives in the restricted aggregation problem P_I^B are the most preferred alternatives in the solution of aggregation problem P_I^C restricted to B.

Remark 3.6. If F satisfies CAL″, then F satisfies CAL′.

Suppose F does not satisfy CAL′: with $C\backslash\{x\} \neq \varnothing$, $F(P_I^{C\backslash\{x\}}) \neq F(P_I^C)\,|_{C\backslash\{x\}}$. Then there are $y \in C$ and $z \in C\backslash\{y\}$ such that y is preferred to z in one of the preferences in $\{F(P_I^{C\backslash\{x\}}),\ F(P_I^C)\}$ but not in the other. Without loss of generality, suppose y is preferred to z in $F(P_I^C)$. Let $B = \{y, z\}$. By CAL″, ${}^1F(P_I^B) = {}^1(F(P_I^C)\,|_B) = \{y\}$. On the other hand, by CAL″, ${}^1F(P_I^B) = {}^1(F(P_I^{C\backslash\{x\}})\,|_B) \neq \{y\}$: contradiction.

Remark 3.7. If F satisfies CAL′, then ${}^1F$ satisfies CAL.

Suppose ${}^1F(P_I^C)\backslash\{x\} \neq \varnothing$. If $x \in {}^1F(P_I^C)$, then, by CAL′, $F(P_I^{C\backslash\{x\}}) = F(P_I^C)\,|_{C\backslash\{x\}}$. As a result, ${}^1F(P_I^{C\backslash\{x\}}) = {}^1(F(P_I^C)\,|_{C\backslash\{x\}}) = {}^1F(P_I^C)\backslash\{x\}$. And if $x \notin {}^1F(P_I^C)$, then, by CAL′, $F(P_I^{C\backslash\{x\}}) = F(P_I^C)\,|_{C\backslash\{x\}})$. Therefore, ${}^1F(P_I^{C\backslash\{x\}}) = {}^1(F(P_I^C)\,|_{C\backslash\{x\}}) = {}^1F(P_I^C) = {}^1F(P_I^C)\backslash\{x\}$.

4. Results

Proposition 4.1. A social choice function f on L satisfies UNA, CAL and CIN if and only if f has a hierarchy of dictators.

Proposition 4.2. If a social choice function f on T satisfies UNA, CAL and CIN then f has a hierarchy of weak dictators.

Proposition 4.3. A social choice rule f on L satisfies UNA, CAL and CIN if and only if f has a hierarchy of dictators.

Proposition 4.4. If a social choice rule f on T satisfies UNA, CAL and CIN then f has a hierarchy of weak dictators.

Proposition 4.5. A social welfare function F on L satisfies UNA′, CAL′ and CIN′ if and only if F has a hierarchy of dictators.

Proposition 4.6. If a social welfare function F on T satisfies UNA′, CAL′ and CIN′ then F has hierarchy of weak dictators.

5. Proofs

For a singleton $\{x\}$, x will be in general written instead of $\{x\}$. Hence, (x, y, z) abbreviates the preference $(\{x\}, \{y\}, \{z\})$. In addition, if f is a social choice function,

$f(P_I^C)$ will be considered an element rather than a set, so "$f(P_I^C) = x$" will be written instead of "$f(P_I^C) = \{x\}$". For a given social choice function f on L, "$i \rightarrow_{xy} j$" abbreviates "for all $P_I^C \in L$ such that $I = \{i, j\}$ and $C = \{x, y\}$, ${}^1P_i^C = x$ implies $f(P_I^C) = x$", whereas "$i \rightarrow j$" abbreviates "for all $x \in A$ and $y \in A \backslash \{x\}$, $i \rightarrow_{xy} j$".

Lemma 5.1. Let f be a social choice function on L that satisfies UNA and CAL. With $I = \{i, j\}$ and $C = \{x, y, z\}$, let $P_I^C \in L$ satisfy $P_i^C = (x, z, y)$ and $P_j^C = (y, x, z)$. Then $f(P_I^C) = x$ implies: (i) $i \rightarrow_{xy} j$ and $i \rightarrow_{xz} j$; and (ii) $i \rightarrow j$.

Proof. Suppose $f(P_I^C) = x$. (i) By CAL, $f(P_I^{C\backslash\{z\}}) = x$. Given this and UNA, $i \rightarrow_{xy} j$. To prove that $i \rightarrow_{xz} j$, consider $Q_I^C \in L$ such that $Q_i^C = (x, y, z)$ and $Q_j^C = (y, z, x)$. If $f(Q_I^C) = z$, then, by CAL, $f(Q_I^{C\backslash\{x\}}) = z$, contradicting UNA. If $f(Q_I^C) = y$, then, by CAL, $f(Q_I^{C\backslash\{z\}}) = y$, which contradicts $i \rightarrow_{xy} j$. Consequently, $f(Q_I^C) = x$. In view of this, by CAL, $f(Q_I^{C\backslash\{y\}}) = x$. This and UNA imply $i \rightarrow_{xz} j$. (ii) Given (i), it must be shown that $i \rightarrow_{yz} j$, $i \rightarrow_{yx} j$, $i \rightarrow_{zx} j$, and $i \rightarrow_{zy}$. Step 1: $i \rightarrow_{yz} j$ and $i \rightarrow_{yx} j$. Let $R_I^C \in L$ be obtained from P_I^C by applying the permutation $\pi(x) = y$, $\pi(z) = x$ and $\pi(y) = z$, so $R_i^C = (y, x, z)$ and $R_j^C = (z, y, x)$. If $f(R_I^C) = x$, then, by CAL, $f(R_I^{C\backslash\{z\}}) = x$, contradicting UNA. If $f(R_I^C) = z$, then, by CAL, $f(R_I^{C\backslash\{y\}}) = z$, which contradicts $i \rightarrow_{xz} j$. Therefore, $f(R_I^C) = y$. Since the permutation π transforms P_I^C into R_I^C, the application of π to (i) yields $i \rightarrow_{yz} j$ and $i \rightarrow_{yx} j$. Step 2: $i \rightarrow_{zx} j$ and $i \rightarrow_{zy} j$. Let $S_I^C \in L$ be obtained from P_I^C by applying the permutation $\pi'(x) = z$, $\pi'(y) = x$ and $\pi'(z) = y$. If $f(S_I^C) = z$, then, by CAL, $f(S_I^{C\backslash\{x\}}) = z$, contradicting UNA. If $f(S_I^C) = x$, then, by CAL, $f(S_I^{C\backslash\{z\}}) = x$, which contradicts $i \rightarrow_{yx} j$. As a result, $f(S_I^C) = z$. Given that the permutation π' transforms P_I^C into S_I^C, the application of π' to (i) yields $i \rightarrow_{zx} j$ and $i \rightarrow_{zy} j$.■

Lemma 5.2. Let f be a social choice function on L that satisfies UNA and CAL. With $I = \{i, j\}$ and $C = \{x, y, z\}$, let $P_I^C \in L$ satisfy $P_i^C = (x, z, y)$ and $P_j^C = (y, x, z)$. Then $f(P_I^C) = y$ implies: (i) $j \rightarrow_{yx} i$ and $j \rightarrow_{yz} i$; and (ii) $j \rightarrow i$.

Proof. Suppose $f(P_I^C) = y$. (i) By CAL, $f(P_I^{C\backslash\{z\}}) = y$. Given this and UNA, $j \rightarrow_{yx} i$. On the other hand, $f(P_I^C) = y$ and CAL also imply $f(P_I^{C\backslash\{x\}}) = y$. Combining this with UNA, $j \rightarrow_{yz} i$. (ii) Let $R_I^C \in L$ be obtained from P_I^C by applying the permutation $\pi(x) = y$, $\pi(z) = x$ and $\pi(y) = z$. If $f(R_I^C) = x$, then, by CAL, $f(R_I^{C\backslash\{z\}}) = x$, which contradicts UNA. If $f(R_I^C) = y$, then, by applying π to the profile P_I^C in Lemma 5.1, $i \rightarrow j$, which contradicts $j \rightarrow_{yx} i$. Accordingly, $f(R_I^C) = z$. Since the permutation π transforms P_I^C into R_I^C, the application of π to (i) yields $j \rightarrow_{zy} i$ and $j \rightarrow_{zx} i$. Finally, let $S_I^C \in L$ be obtained from P_I^C by applying the permutation $\pi'(x) = z$, $\pi'(y) = x$ and $\pi'(z) = y$. If $f(S_I^C) = y$, then, by CAL, $f(S_I^{C\backslash\{x\}}) = y$, contradicting UNA. If $f(S_I^C) = z$, then, by applying π' to the profile P_I^C in

Lemma 5.1, $i \to j$, which contradicts $j \to_{yx} i$. In sum, $f(S_I^C) = x$. Since the permutation π' transforms P_I^C into S_I^C, the application of π' to (i) yields $j \to_{xz} i$ and $j \to_{xy} i$.■

Lemma 5.3. Let f be a social choice function on L that satisfies UNA and CAL. Then, for all $i \in N$, $j \in N\backslash\{i\}$ and $k \in N\backslash\{i, j\}$: (i) either $i \to j$ or $j \to i$; and (ii) if CIN holds, then $i \to j$ and $j \to k$ imply $i \to k$.

Proof. (i) With $I = \{i, j\}$ and $C = \{x, y, z\}$, let $P_I^C \in L$ satisfy $P_i^C = (x, z, y)$ and $P_j^C = (y, x, z)$. If $f(P_I^C) = x$, then, by Lemma 5.1, $i \to j$. If $f(P_I^C) = y$, then, by Lemma 5.2, $j \to i$. And if $f(P_I^C) = z$, then, by CAL, $f(P_I^{C\backslash\{y\}}) = z$, contradicting UNA. (ii) Suppose $i \to j$ and $j \to k$. By (i), either $i \to k$ or $k \to i$. Suppose $k \to i$. With $I = \{i, j, k\}$ and $C = \{x, y, z\}$, let $P_I^C \in L$ satisfy $P_i^C = (x, y, z)$, $P_j^C = (y, z, x)$ and $P_k^C = (z, x, y)$. If $f(P_I^C) = x$, by CAL, $f(P_I^{C\backslash\{y\}}) = x$ and, by CIN, $f(P_{I\backslash\{j\}}^{C\backslash\{y\}}) = x$, contradicting $k \to i$. If $f(P_I^C) = y$, by CAL, $f(P_I^{C\backslash\{z\}}) = y$ and, by CIN, $f(P_{I\backslash\{k\}}^{C\backslash\{z\}}) = y$, contradicting $i \to j$. If $f(P_I^C) = z$, by CAL, $f(P_I^{C\backslash\{x\}}) = z$ and, by CIN, $f(P_{I\backslash\{i\}}^{C\backslash\{x\}}) = z$, contradicting $j \to k$.■

Lemma 5.4. Let f be a social choice function on L that satisfies CAL and CIN. For $i \in N$, define $I(i) = \{j \in N\colon i \to j\}$. Then, for all $P_I^C \in L$ such that C has two elements and $I \subseteq I(i) \cup \{i\}$, $f(P_I^C) = {}^1P_i^C$.

Proof. Let $C = \{x, y\}$ be a choice set. If I has two members, with $I = \{i, j\}$, then $f(P_I^C) = {}^1P_i^C$ follows directly from the fact that $i \to j$. Taking this result as the base case of an induction argument, choose $r \geq 3$ and assume that, for each society $I \subseteq I(i) \cup \{i\}$ of size $s < r$, $f(P_I^C) = {}^1P_i^C$. To show that $f(P_I^C) = {}^1P_i^C$ for any society $I \subseteq I(i) \cup \{i\}$ of size r, choose $I \subseteq I(i) \cup \{i\}$ having r members and $P_I^C \in L$. Suppose $y = f(P_I^C) \neq {}^1P_i^C = x$. Case 1: for some $j \in I\backslash\{i\}$, $P_j^C = P_i^C$. By CIN, $f(P_{I\backslash\{j\}}) = f(P_I^C) \neq {}^1P_i^C$, which contradicts the induction hypothesis. Case 2: for all $j \in I\backslash\{i\}$, $P_j^C \neq P_i^C$. Choose $z \in A\backslash C$ and $j \in I\backslash\{i\}$. With $B = C \cup \{z\}$, let $Q_I^B \in L$ satisfy $Q_i^B = (x, z, y)$, $Q_j^B = (z, y, x)$ and, for all $k \in I\backslash\{i, j\}$, $Q_k^B = (y, x, z)$. If $f(Q_I^B) = x$, then, by CAL, $f(Q_I^C) = x$. Since $Q_I^C = P_I^C$, $f(Q_I^C) = x$ contradicts the hypothesis that $f(P_I^C) \neq x$. If $f(Q_I^B) = y$, then, by CAL, $f(Q_I^{C\backslash\{x\}}) = y$. By case 1, it must be that $f(Q_I^{C\backslash\{x\}}) = {}^1Q_i^{C\backslash\{x\}} = z$: contradiction. If $f(Q_I^B) = z$, then, by CAL, $f(Q_I^{C\backslash\{y\}}) = z$. By case 1, it must be that $f(Q_I^{C\backslash\{y\}}) = {}^1Q_i^{C\backslash\{y\}} = x$: contradiction.■

Proof of Proposition 4.1. "⇐" It can be easily verified that if f has a hierarchy of dictators, then f satisfies UNA, CAL and CIN. "⇒" By Lemma 5.3, the n members of N can be arranged in a sequence $(i_1, \ldots, i_n)$ such that, for all $r \in \{1, \ldots, n-1\}$, $i_r \to i_{r+1}$. It will be shown that this sequence defines a hierarchy of dictators. To this end, choose $P_I^C \in L$. Let i be the member of I appearing first in the sequence $(i_1, \ldots, i_n)$. Therefore, for all $j \in I\backslash\{i\}$, $i \to j$. The proof amounts to showing that $f(P_I^C) = {}^1P_i^C$. Suppose not: $f(P_I^C) = y \neq x = {}^1P_i^C$. By UNA, I has more than one member. With $D = \{x, y\}$, by CAL, $f(P_I^D) = y \neq {}^1P_i^D$. This contradicts Lemma 5.4.■

Proof of Proposition 4.2. By Proposition 4.1, f restricted to L has a hierarchy of dictators $(i_1, \dots, i_n)$. To see that $(i_1, \dots, i_n)$ is a hierarchy of weak dictators of f, choose $P_I^C \in T$ and let i be the member of I appearing first in the list $(i_1, \dots, i_n)$. It must be shown that $f(P_I^C) \in {}^1P_i^C$. Suppose not: $f(P_I^C) = x \notin {}^1P_i^C$. Choose $y \in {}^1P_i^C$. Letting $B = \{x, y\}$, by CAL, $f(P_I^C) = x$ implies $f(P_I^B) = x$. Choose $z \in A\backslash B$. With $C = \{x, y, z\}$, let $Q_I^C \in T$ satisfy $Q_i^C = (y, z, x)$ and, for all $j \in I\backslash\{i\}$, z is the least preferred object in Q_j^C, so $Q_j^C \in \{(xy, z), (x, y, z), (y, x, z)\}$. If $f(P_I^C) = y$, then, by CAL, $f(Q_I^B) = y$. Since $Q_I^B = P_I^B$, $f(Q_I^B) = y$ contradicts $f(P_I^B) = x$. If $f(P_I^C) = x$, then, by CAL, $f(Q_I^{\{x,z\}}) = x$. But $Q_I^{\{x,z\}} \in L$, so it must be that $f(Q_I^{\{x,z\}}) = {}^1Q_I^{\{x,z\}} = z$: contradiction. If $f(P_I^C) = z$, then, by CAL, $f(Q_I^{\{y,z\}}) = z$. Since $Q_I^{\{y,z\}} \in L$, $f(Q_I^{\{y,z\}}) = {}^1Q_I^{\{y,z\}} = y$: contradiction.■

Proof of Proposition 4.3. "$\Leftarrow$" It can be easily verified that if f has a hierarchy of dictators, then f satisfies UNA, CAL and CIN. "$\Rightarrow$" By Proposition 4.2, it is enough to show that f is a social choice function. Suppose not: for some $P_I^C \in L$, $f(P_I^C)$ has at least two members. Let $B = \{x, y\} \subseteq f(P_I^C)$. By CAL, $f(P_I^B) = f(P_I^C) \cap B = B$. By UNA, there is a non-empty $J \subset I$ such that, for all $i \in J$, ${}^1P_I^B = x$ and, for all $i \in I\backslash J$, ${}^1P_I^B = y$. Let $z \in A\backslash B$ and $D = \{x, y, z\}$. Define $Q_I^D \in L$ to be such that, for all $i \in J$, $Q_i^D = (x, z, y)$ and, for all $i \in I\backslash J$, $Q_i^D = (z, y, x)$. If $f(Q_I^D) = \{x\}$ or $f(Q_I^D) = \{x, z\}$, then, by CAL, $f(Q_I^B) = f(Q_I^D)\backslash\{z\} = \{x\}$. But $Q_I^B = P_I^B$ implies $f(Q_I^B) = f(P_I^B) = B$: contradiction. If $f(Q_I^D) = \{y\}$ or $f(Q_I^D) = \{y, z\}$, then, by CAL, $f(Q_I^B) = f(Q_I^D)\backslash\{z\} = \{y\}$. But $Q_I^B = P_I^B$ implies $f(Q_I^B) = f(P_I^B) = B$: contradiction. If $f(Q_I^D) = D$, then, by CAL, $f(Q_I^{\{y,z\}}) = f(Q_I^D)\backslash\{x\} = \{y, z\}$, contradicting UNA. If $f(Q_I^D) = \{x, y\}$, then, by CAL, $f(Q_I^{\{y,z\}}) = f(Q_I^D)\backslash\{x\} = \{y\}$, contradicting UNA. Summarizing, $f(Q_I^D) = \{z\}$. By CAL, $f(Q_I^{\{x,z\}}) = f(Q_I^D)\backslash\{y\} = \{z\}$.

Let $R_I^D \in L$ satisfy, for all $i \in J$, $R_i^D = (x, y, z)$ and, for all $i \in I\backslash J$, $R_i^D = (y, z, x)$. If $f(R_I^D) = \{z\}$, then, by CAL, $f(R_I^{\{y,z\}}) = f(R_I^D)\backslash\{x\} = \{z\}$, contradicting UNA. If $f(R_I^D) = \{x\}$ or $f(R_I^D) = \{x, z\}$, then, by CAL, $f(R_I^B) = f(R_I^D)\backslash\{z\} = \{x\}$. Since $R_I^B = P_I^B$, $f(R_I^B) = f(P_I^B) = B$: contradiction. If $f(R_I^D) = \{y\}$ or $f(R_I^D) = \{y, z\}$, then, by CAL, $f(R_I^B) = f(R_I^D)\backslash\{z\} = \{y\}$. Since $R_I^B = P_I^B$, $f(R_I^B) = f(P_I^B) = B$: contradiction. If $f(R_I^D) = \{x, y\}$, then, by CAL, $f(R_I^{\{x,z\}}) = f(R_I^D)\backslash\{y\} = \{x\}$. It then follows from $R_I^{\{x,z\}} = Q_I^{\{x,z\}}$ that $f(R_I^{\{x,z\}}) = f(Q_I^{\{x,z\}}) = \{z\}$: contradiction. Finally, if $f(R_I^D) = D$, then, by CAL, $f(R_I^{\{x,z\}}) = f(R_I^D)\backslash\{y\} = \{x, z\}$. But $R_I^{\{x,z\}} = Q_I^{\{x,z\}}$ implies $f(R_I^{\{x,z\}}) = f(Q_I^{\{x,z\}}) = \{z\}$: contradiction.■

Proof of Proposition 4.4. By Proposition 4.3, f restricted to L has a hierarchy of dictators $(i_1, \ldots, i_n)$. Choose $P_I^C \in T$ and let i be the member of I appearing first in the list $(i_1, \ldots, i_n)$. It must be shown that $f(P_I^C) \subseteq {}^1P_i^C$. Suppose not: for some $y \in f(P_I^C)$, $y \notin {}^1P_i^C$. Choose $x \in {}^1P_i^C$. Letting $B = \{x, y\}$, by CAL, $f(P_I^B) = f(P_I^C) \cap B$. Hence, either $f(P_I^B) = \{y\}$ or $f(P_I^B) = \{x, y\}$. With $z \in A\backslash B$ and $D = \{x, y, z\}$, let $Q_I^D \in L$ satisfy $Q_i^D = (x, z, y)$ and, for all $j \in I\backslash\{i\}$, Q_j^D is obtained from P_j^I by making z the most preferred alternative. If $f(Q_I^D) = \{x\}$ or $f(Q_I^D) = \{x, z\}$, then, by CAL, $f(Q_I^B) = f(Q_I^D)\backslash\{z\} = \{x\}$. Since $Q_I^B = P_I^B$, $f(Q_I^B) = f(P_I^B) \neq \{x\}$: contradiction. If $f(Q_I^D) \in \{\{y\}, \{y, z\}, \{x, y\}, \{x, y, z\}\}$, then, by CAL, $f(Q_I^{\{y,z\}}) = f(Q_I^D)\backslash\{x\} \neq \{z\}$, which contradicts UNA. Accordingly, $f(Q_I^D) = \{z\}$. Therefore, by CAL, $f(Q_I^{\{x,z\}}) = f(Q_I^D)\backslash\{y\} = \{z\}$. Since $Q_I^{\{x,z\}} \in L$, $f(Q_I^{\{x,z\}}) = \{z\}$ contradicts the fact that f restricted to L has the hierarchy of dictators $(i_1, \ldots, i_n)$.■

Proof of Proposition 4.5. "⇐" It can be easily verified that if F has a hierarchy of dictators, then F satisfies UNA′, CAL′ and CIN′. "⇒" Since F satisfies UNA′ and CIN′, ${}^1F$ satisfies UNA and CIN. By Remark 3.7, ${}^1F$ satisfies CAL. By Proposition 4.3, ${}^1F$ has some hierarchy of dictators $(i_1, \ldots, i_n)$. The proof amounts to showing that $(i_1, \ldots, i_n)$ is also a hierarchy of dictators in F. Suppose not: for some $P_I^C \in L$, $F(P_I^C) \neq P_i^C$, where i is the member of I appearing first in $(i_1, \ldots, i_n)$. Let r be the smallest positive integer such that ${}^rF(P_I^C) \neq {}^rP_i^C$. If $r = 1$, define $B = C$. If $r \geq 2$, define $B = C\backslash({}^1P_i^C \cup \ldots \cup {}^{r-1}P_i^C)$. B is not a singleton, because this would imply $F(P_I^C) = P_i^C$. Consider P_I^B. Since, for all $t \in \{1, \ldots, r-1\}$, ${}^tF(P_I^C) = {}^tP_i^C$, ${}^1F(P_I^B) \subseteq B$. By CAL′, $F(P_I^B) = F(P_I^C)\big|_B$. Accordingly, ${}^1F(P_I^B) = {}^rF(P_I^C)$. Since ${}^1F$ has the hierarchy of dictators $(i_1, \ldots, i_n)$, ${}^1F(P_I^B) = {}^1P_i^B = {}^rP_i^C$: contradiction.■

Proof of Proposition 4.6. By Proposition 4.5, F restricted to L has a hierarchy of dictators $(i_1, \ldots, i_n)$. Choose $P_I^C \in T$ and let i be the member of I appearing first in the list $(i_1, \ldots, i_n)$. It must be shown that, for all $x \in C$ and $y \in C\backslash\{x\}$, if x is preferred to y in P_i^C, then x is preferred to y in $F(P_I^C)$. Suppose not: x is preferred to y in P_i^C but x is not preferred to y in $F(P_I^C)$. With $B = \{x, y\}$, by CAL′, $F(P_I^B) = F(P_I^C)\big|_B$. Since F satisfies UNA′ and CIN′, ${}^1F$ satisfies UNA and CIN. By Remark 3.7, ${}^1F$ satisfies CAL. In consequence, ${}^1F$ has a hierarchy of dictators $(j_1, \ldots, j_n)$. Given that F restricted to L has the hierarchy of dictators $(i_1, \ldots, i_n)$, it must be that $(j_1, \ldots, j_n) = (i_1, \ldots, i_n)$. As a result, x preferred to y in P_i^B implies ${}^1F(P_I^C) \cap B = \{x\}$. Hence, $F(P_I^B) = F(P_I^C)\big|_B$ implies ${}^1(F(P_I^C)\big|_B) \cap B = \{x\}$. In view of this, x is preferred to y in $F(P_I^C)$: contradiction.■

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