

We cannot choose among ourselves consistently

Antonio Quesada[†]

Departament d'Economia, Universitat Rovira i Virgili, Avinguda de la Universitat 1, 43204 Reus, Spain

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Abstract

The problem of selecting an individual from a given set of individuals is considered when the choice depends on the preferences of the individuals over themselves. The selected individual can be viewed as a representative or as a manager of the group. It is shown that no unanimous rule to choose the individual can be consistent in the sense that the selected individual is still chosen after removing an individual that has not been selected. It is also shown that the same incompatibility between unanimity and consistency arises when the rule can choose subsets of individuals rather than only individuals.

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[†] E-mail address: aqa@urv.cat. Financial support from the Spanish *Ministerio de Educación y Ciencia* (research project SEJ2007-67580-C02-01) is gratefully acknowledged.

1. Introduction

It is not difficult to identify situations in which the individuals from a certain group or collective must choose one of their members to perform some task or to hold some post. For example, the speaker or president of a parliamentary chamber is typically chosen by the members of the chamber among themselves. Other instances of individuals chosen among peers are the chairman of a board or a committee, a sports team captain, the president of a residents' association, the president of a scientific association, a club's president, or the head of a university department.

This type of choice problem does not only arise within the context of social, political or economic institutions. When someone has a date with someone else, the problem of who pays the dinner is another instance of the choosing among ourselves problem. Even every individual faces this same problem when making a decision, since each such decision could be regarded as a collective decision in which the person takes into account the person he was, the person he is and the person he will be. In view of this, an individual with a strong preference for the present will tend to base his decision on the person he is, so it can be interpreted that the individual chooses (the preferences of) his present self to make the decision. On the other hand, an individual concerned with the future consequences of his decision may be considered to rely on his future self to make the decision. And an individual with a strict vision of who he is, will probably act as if he delegated the decision to his past self.

Social choice functions constitute a way of formalizing collective decision problems. Given a set of alternatives and a set of individuals, a social choice function takes the preferences of the individuals over the alternatives and outputs an alternative, representing the alternative chosen by the collective with the given preferences. The Gibbard (1973) - Satterthwaite (1975) theorem is arguably the fundamental results for social choice functions. It essentially says that the only social choice functions in which individuals do not have an incentive to lie about their preferences are those in which some individual always dictates the social choice.

In the standard social choice framework the set of individuals and the set of alternatives are conceptually different sets. This paper considers social choice functions for the particular case in which the set of alternatives coincides with the set of individuals. Two properties are suggested: unanimity and consistency. Unanimity states that if some individual i is the one that all the individuals prefer most, then i is the chosen individual. Consistency asserts that if i is the chosen individual when the group of individuals I has

preferences P , then i must remain the chosen individual in any subgroup J of I that includes i and where the preferences of the members of J are taken from P .

In general, consistency is the principle according to which the solution s of a problem should be (or, at least, should be related to) the solution of a restriction of the problem in which the s is still available. Instances of the consistency principle can be found in many decision-making fields. In cooperative game theory, as a property of solutions; see Thomson (1990). In non-cooperative game theory, as an axiom to characterize equilibrium concepts and as a solution for extensive form games (subgame perfection); see Peleg and Tijs (1996) and Selten (1975). In axiomatic bargaining theory, it takes the form of the independence of irrelevant alternatives condition that characterizes the Nash (1950) bargaining solution. In social choice theory, the independence of irrelevant alternatives condition in Arrow's (1963, p. 97) theorem can also be viewed as a requirement of consistency relating the aggregation of preferences over n alternatives to the aggregation over just two. And in the theory of the allocation of indivisible goods, consistency contributes to generate hierarchical allocation structures; see Ergin (2000).

This note shows that no unanimous social choice function can be consistent when there are at least three individuals, so the adoption of a social choice function that respects unanimity implies that consistency must fail. Whereas unanimity is an axiom that is difficult to reject, why should consistency be adopted? Consistency can be regarded as a requirement of stability: decisions should not be reconsidered unless there are reasons powerful enough. Consistency presumes that removing a non-selected individual is not enough reason. Consider, for instance, one of the most powerful legislatures in the world: the European Parliament. The President of the European Parliament (its speaker) is chosen among its 736 members. Imagine that one of its members dies or resigns. Should the President be elected again? Consistency says that if the member is not the President, then it should not. Hence, under an inconsistent social choice function, any change in the set of individuals may lead to a reconsideration of the chosen individual. For political and economic institutions, this may turn the institution dysfunctional.

The tension between unanimity and consistency is not limited to social choice functions. It is shown that this tension persists when subsets of individuals may be chosen. In this case, no social correspondence satisfies unanimity and the strong consistency condition stating the following. Suppose I^* is the set of individuals chosen from an initial set I of individuals. Let J be a subset of I . Then the set of individuals chosen in $I \setminus J$ must be $I^* \setminus J$, provided $I^* \setminus J \neq \emptyset$: the solution when the members of J are removed is obtained by removing the members of J from the initial solution.

2. Definitions

Members of the set $N = \{1, 2, \dots, n\}$ are names for individuals. A society is a finite non-empty subset of N having at least two elements. For society I , a preference p on I is a sequence (I_1, \dots, I_m) of societies such that: (i) for all $r \in \{1, \dots, m\}$ and $s \in \{1, \dots, m\} \setminus \{r\}$, $I_r \cap I_s = \emptyset$; and (ii) $I_1 \cup \dots \cup I_m = I$. The interpretation is that, for all $r \in \{1, \dots, m\}$, each member of I_r is indifferent to the rest of members of I_r and, for $r < m$, each such member is preferred to every member of $I_{r+1} \cup \dots \cup I_m$. For preference $p = (I_1, \dots, I_m)$ on I and $r \in \{1, 2, \dots, m\}$, I_r designates the r th element in the sequence p . For instance, I_1 represents the set of most preferred individuals in p . A preference $p = (I_1, \dots, I_m)$ on I is strict if, for all $r \in \{1, \dots, m\}$, I_r has only one element.

Let T_I be the set of preferences that can be defined on I . A preference profile for society I is a function $P_I : I \rightarrow T_I$ assigning a preference on I to each member of I . Let T be the set of all preference profiles P_I such that I is a society. For $P_I \in T$ and $i \in I$, P_i will abbreviate $P_I(i)$. A preference profile P_I such that, for all $i \in I$, P_i is a strict preference is called a strict preference profile. Let L be the set of all strict preference profiles P_I such that I is a society. The restriction P_J of $P_I \in T$ to $J \subseteq I$ is the preference profile $Q_J : J \rightarrow T_J$ such that, for all $i \in J$, $Q_J(i) = P_I(i)$.

Definition 2.1. A social choice function on $X \subseteq T$ is a mapping $f : X \rightarrow N$ such that, for all $P_I \in X$, $f(P_I) \in I$.

A social choice function takes as input the preferences of all the members of any given society I over themselves and outputs a member of the society, who will be interpreted as the individual chosen by society I .

Definition 2.2. The set $X \subseteq T$ is closed if, for all $P_I \in X$ and society $J \subseteq I$, $P_J \in X$.

Definition 2.3. A social choice function f on a closed $X \subseteq T$ is consistent if, for all $P_I \in X$ and $J \subseteq I$, if $f(P_I) \in J$, then $f(P_J) = f(P_I)$.

Consistency holds that if i is chosen by society I , then i is also chosen by every subsociety of I containing i . Consistency can be equivalently formulated in terms of expansions, rather than contractions, of a society: $f(P_I) = i$ and $J \subseteq NI$ imply $f(P_{I \cup J}) \in J \cup \{i\}$. This suggests that consistent choice can be viewed as a reference-dependent choice; see Bossert and Sprumont (2009).

Definition 2.4. A social choice function on $X \subseteq T$ is unanimous when, for all $P_I \in X$, if there is $i \in I$ such that, for all $j \in I$, ${}^1P_j = \{i\}$, then $f(P_I) = i$.

3. An impossibility result for social choice functions

Proposition 3.1. Let $n \geq 3$ and $X \subseteq T$ be closed. If $L \subseteq X$, then there is no unanimous and consistent social choice function on X .

Proof. Define individual k to have veto power if, for each preference profile $P_I \in X$ such that $k \in I$, it is not the case that k is preferred to $f(P_I)$ in P_k . Hence, k has veto power if k never prefers himself to the individual chosen by any society to which k belongs. Consider any preference profile $Q_N \in L$ such that, for all $k \in N$, ${}^1Q_k = \{k\}$. Let $f(Q_N) = i$.

- Step 1: i has veto power. Choose $P_I \in L$ such that $i \in I$. Letting $f(P_I) = k$, it must be shown that it is not the case that i is preferred to k in P_i . To this end, suppose otherwise: i is preferred to k in P_i . With $J = \{i, k\}$, consider P_J . By consistency, $f(P_I) = k$ implies $f(P_J) = k$. This contradicts unanimity if i is preferred to k in P_k . Therefore, k is preferred to i in P_k . By consistency, $f(Q_N) = i$ implies $f(Q_J) = i$. Since $P_J = Q_J$, $k = f(P_J) = f(Q_J) = i$: contradiction.

- Step 2: for all $P_I \in L$ and $j \in \Lambda\{i\}$ such that $i \in I$, if ${}^1P_i = \{j\}$, ${}^2P_i = \{i\}$ and j is preferred to i in P_j , then $f(P_I) = j$. By step 1, i has veto power, so $f(P_I) \in \{i, j\}$. To prove that $f(P_I) = j$, suppose otherwise: $f(P_I) = i$. With $J = \{i, j\}$, by consistency, $f(P_J) = i$. This contradicts unanimity because ${}^1P_J(i) = {}^1P_J(j) = \{j\}$.

To conclude the proof, choose $j \in \mathcal{M}\{i\}$ and $k \in \mathcal{M}\{i, j\}$. Define $I = \{i, j, k\}$. Let $R_I \in L$ satisfy ${}^1R_i = \{j\}$, ${}^2R_i = \{i\}$, ${}^1R_j = \{j\}$, and ${}^1R_k = \{k\}$. By step 2, $f(R_I) = j$. By consistency, $f(R_{\{j,k\}}) = j$. Let $S_I \in L$ be obtained from R_I by just permuting j and k in R_i , so ${}^1S_i = \{k\}$ and ${}^2S_i = \{i\}$. By step 2, $f(S_I) = k$. By consistency, $f(S_{\{j,k\}}) = k$. Since $R_{\{j,k\}} = S_{\{j,k\}}$, it must be that $f(R_{\{j,k\}}) = f(S_{\{j,k\}})$: contradiction. ■

Proposition 3.1 asserts that, when there are at least three individuals, if a social choice function defined for a set of preference profiles containing all the strict preference profiles is unanimous, then the social choice function cannot be consistent. Hence, unlike the Gibbard-Satterthwaite theorem, it is not necessary to exclude the possibility that individuals are indifferent.

It is also worth noticing that the proof of Proposition 3.1 uses preference information concerning only two most preferred individuals. Therefore, the incompatibility between unanimity and consistency persists when individuals are just asked to indicate which are their two most preferred individuals. For instance, suppose that paper ballots convey preference information and that each individual has to write the name of two individuals in his ballot. Then the proof of Proposition 3.1 shows that there is no way of using the information in the paper ballots to select an individual consistently and unanimously.

4. Two impossibility results for social choice correspondences

It will next be shown that the incompatibility between unanimity and consistency also persists when subsets of individuals can be chosen.

Definition 4.1. With N^* being the set of societies of N , a social choice correspondence on $X \subseteq T$ is a mapping $f: X \rightarrow N^*$ such that, for all $P_I \in X$, $f(P_I) \subseteq I$.

A social choice correspondence takes as input the preferences of all the members of any given society I over themselves and outputs a subset $f(P_I)$ of members of the society, interpreted as the set of individual chosen by society I .

Definition 4.2. A social choice correspondence f on a closed $X \subseteq T$ is consistent if, for all $P_I \in X$ and $J \subseteq I$, if $f(P_I) \subseteq J$, then $f(P_J) = f(P_I)$.

Consistency of a social choice correspondence requires that, if the individuals chosen by society I are taken from J , then the individuals chosen by J should be selected among those chosen by I .

Definition 4.3. A social choice correspondence on $X \subseteq T$ is unanimous when, for all $P_I \in X$, if there is $i \in I$ such that, for all $j \in I$, $^1P_j = \{i\}$, then $f(P_I) = \{i\}$.

As in the social choice function case, unanimity defines a situation in which the choice of individuals should be trivial: if all the members of the society agree in having i as the most preferred individual, then i should be the only chosen individual. This can be justified by a principle of Pareto efficiency: if all the individuals prefer i to j , then j cannot be chosen.

Definition 4.4. A social choice correspondence on $X \subseteq T$ is resolute when, for all $P_I \in X$, $f(P_I) \subset I$.

Resoluteness means that societies always discard some member. In other words, it is not admissible to choose everybody. Interpreting $f(P_I)$ as the set of representatives acting on behalf of society I , resoluteness implies that I is not a valid set of representatives for I .

Proposition 4.5. Let $n \geq 3$ and $X \subseteq T$ be closed. If $L \subseteq X$, then there is no unanimous, resolute and consistent social choice correspondence on X .

Proof. It will be shown first that, for all $P_I \in L$, $f(P_I)$ is a singleton. Suppose not: for some $P_I \in L$, $f(P_I)$ has at least two members, i and j . Letting $J = \{i, j\}$, by consistency, $f(P_I) \subseteq J$ implies $f(P_J) = J$, which contradicts resoluteness. Since, for all $P_I \in L$, $f(P_I)$ is a singleton, f restricted to L can be considered a social choice function on L . On that domain, the property “ $f(P_I) \subseteq J$ implies $f(P_J) = f(P_I)$ ” that defines consistency of f as a social choice correspondence is equivalent to the property “ $f(P_I) \in J$ implies $f(P_J) = f(P_I)$ ” that defines consistency of a social choice function. On the other hand, unanimity as defined for social choice functions is equivalent to unanimity as defined for social choice correspondences. Summarizing, f restricted to L is a unanimous and consistent social choice function, contradicting Proposition 3.1. ■

Definition 4.6. A social choice correspondence f on a closed $X \subseteq T$ is strongly consistent if, for all $P_I \in X$ and $J \subseteq I$, $f(P_I) \cap J \neq \emptyset$ implies $f(P_J) = f(P_I) \cap J$.

Consistency deals with the case in which, after society I losses some of its members, all the chosen individuals $f(P_I)$ remain. For this case, consistency ensures that there is no need to reconsider the chosen set: if J is obtained from I by removing members not in $f(P_I)$, then $f(P_I)$ is still the set of chosen individuals from J . Strong consistency deals with the case in which J is obtained from I by removing members not in $f(P_I)$ and some (but not all the) members of $f(P_I)$. When some members of $f(P_I)$ are lost, strong consistency requires the chosen members to be the surviving members of $f(P_I)$.

Proposition 4.7. Let $n \geq 3$ and $X \subseteq T$ be closed. If $L \subseteq X$, then there is no unanimous and strongly consistent social choice correspondence on X .

Proof. Since strong consistency implies consistency, by Proposition 4.5, it is enough to show that f is resolute. Suppose not: for some $P_I \in X$, $f(P_I) = I$, where I has at least two members. Choose $i \in I$ and $j \in I \setminus \{i\}$. Letting $J = \{i, j\}$, by consistency, $f(P_J) = J$. By

unanimity, $P_J(i) \neq P_J(j)$. Case 1: ${}^1P_J(i) = \{i\}$. Therefore, ${}^1P_J(j) = \{j\}$. Choose $k \in N \setminus J$. With $K = \{i, j, k\}$, let $Q_K \in X$ satisfy ${}^1Q_i = \{i\}$, ${}^2Q_i = \{k\}$, ${}^1Q_j = \{k\}$, ${}^2Q_j = \{j\}$, and $Q_k = Q_j$. If $f(Q_K) = \{i\}$ or $f(Q_K) = \{i, k\}$, then, by consistency, $f(Q_J) = f(Q_K) \cap J = \{i\}$. But $Q_J = P_J$, so it must be that $f(Q_J) = f(P_J) = J$: contradiction. If $f(Q_K) = \{j\}$ or $f(Q_K) = \{j, k\}$, then, by consistency, $f(Q_J) = f(Q_K) \cap J = \{j\}$. Since $Q_J = P_J$, $f(Q_J) = f(P_J) = J$: contradiction. If $f(Q_K) = \{i, j\}$ or $f(Q_K) = \{i, j, k\}$, then, by consistency, $f(Q_{\{j,k\}}) = f(Q_K) \cap \{j, k\} = \{j\}$, which contradicts unanimity. As a result, $f(Q_K) = \{k\}$. This and consistency imply $f(Q_{\{i,k\}}) = \{k\}$.

Let $R_K \in X$ satisfy ${}^1R_i = \{i\}$, ${}^2R_i = \{j\}$, ${}^1R_j = \{j\}$, ${}^2R_j = \{k\}$, and $R_k = R_j$. If $f(R_K) = \{i\}$ or $f(R_K) = \{i, k\}$, then, by consistency, $f(R_J) = f(R_K) \cap J = \{i\}$. It then follows from $R_J = P_J$ that $f(R_J) = f(P_J) = J$: contradiction. If $f(R_K) = \{j\}$ or $f(R_K) = \{j, k\}$, then, by consistency, $f(R_J) = f(R_K) \cap J = \{j\}$. As $R_J = P_J$, $f(R_J) = f(P_J) = J$: contradiction. If $f(R_K) = \{k\}$ or $f(R_K) = \{i, j, k\}$, then, by consistency, $f(R_{\{j,k\}}) = f(R_K) \cap \{j, k\}$. Accordingly, $k \in f(R_{\{j,k\}})$, contradicting unanimity. As a consequence, it must be that $f(R_K) = \{i, j\}$. Given this, by consistency, $f(R_{\{i,k\}}) = f(R_K) \cap \{i, k\} = \{i\}$. But $R_{\{i,k\}} = Q_{\{i,k\}}$ implies $f(R_{\{i,k\}}) = f(Q_{\{i,k\}}) = \{k\}$: contradiction.

Case 2: ${}^1P_J(i) = \{j\}$. Choose $k \in N \setminus J$. With $K = \{i, j, k\}$, let $Q_K \in X$ satisfy ${}^1Q_i = \{k\}$, ${}^2Q_i = \{j\}$, ${}^1Q_j = \{i\}$, ${}^2Q_j = \{k\}$, and $Q_k = Q_j$. If $f(Q_K) = \{i\}$ or $f(Q_K) = \{i, k\}$, then, by consistency, $f(Q_J) = f(Q_K) \cap J = \{i\}$. Since $Q_J = P_J$, $f(Q_J) = f(P_J) = J$: contradiction. If $f(Q_K) = \{j\}$ or $f(Q_K) = \{j, k\}$, then, by consistency, $f(Q_J) = f(Q_K) \cap J = \{j\}$. It then follows from $Q_J = P_J$ that $f(Q_J) = f(P_J) = J$: contradiction. If $f(Q_K) = \{i, j\}$ or $f(Q_K) = \{i, j, k\}$, then, by consistency, $f(Q_{\{j,k\}}) = f(Q_K) \cap \{j, k\} = \{j\}$, which contradicts unanimity. Summing up, $f(Q_K) = \{k\}$. This and consistency imply $f(Q_{\{i,k\}}) = \{k\}$.

Let $R_K \in X$ satisfy ${}^1R_i = \{j\}$, ${}^2R_i = \{k\}$, ${}^1R_j = \{i\}$, ${}^2R_j = \{j\}$, and $R_k = R_j$. If $f(R_K) = \{i\}$ or $f(R_K) = \{i, k\}$, then, by consistency, $f(R_J) = f(R_K) \cap J = \{i\}$. As $R_J = P_J$, $f(R_J) = f(P_J) = J$: contradiction. If $f(R_K) = \{j\}$ or $f(R_K) = \{j, k\}$, then, by consistency, $f(R_J) = f(R_K) \cap J = \{j\}$. Given $R_J = P_J$, it must be that $f(R_J) = f(P_J) = J$: contradiction. If $f(R_K) = \{k\}$ or $f(R_K) = \{i, j, k\}$, then, by consistency, $f(R_{\{j,k\}}) = f(R_K) \cap \{j, k\}$. Therefore, $k \in f(R_{\{j,k\}})$, contradicting unanimity. All in all, $f(R_K) = \{i, j\}$. Given this, by consistency, $f(R_{\{i,k\}}) = f(R_K) \cap \{i, k\} = \{i\}$. But $R_{\{i,k\}} = Q_{\{i,k\}}$ implies $f(R_{\{i,k\}}) = f(Q_{\{i,k\}}) = \{k\}$: contradiction. ■

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