

Monotonicity + Efficiency + Continuity = Majority

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2nd January 2010

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Abstract

Axioms of monotonicity, efficiency and continuity are shown to characterize the relative majority rule when there are only two alternatives. The absolute majority rule and the relative majority rule in which indifferences are resolved following some given tie-breaking rule are also characterized using those axioms. The strategy followed in these two characterizations consists of: (i) identifying a domain D where the relative majority rule coincides with the characterized rule; and (ii) making the value of elements not in D coincide with the value of some element in D .

Keywords: Social welfare function, majority rules, axiomatic characterization, two alternatives.

JEL Classification: D71

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1. Introduction

The default strategy for the study of individual economic decisions probably consists of framing the problem in a market in order to next look for market equilibria. The default approach to the study of collective decisions in non-market organizations probably consists of invoking some majority rule. One of the difficulties when dealing with the first decision problem is that equilibria may not exist and, even if they exist, they may be not reasonable. The basic disadvantage associated with majority rules is that they may create preference cycles. This drawback can be avoided by restricting decisions to those involving two options. Though this implies some loss of generality, it is worth noticing that decisions over n options can be decomposed into a sequence of decisions over two alternatives, consisting of a given option and the rest of non-discarded options.

There are several characterizations of the weakest majority concept: the relative majority rule. Those by May (1952, p. 682) and Fishburn (1973, p. 58) deal with the case of a fixed set of individuals. The one by Xu and Zhong (2009) holds the preferences fixed and allows the set of individuals to vary. Those by Aşan and Sanver (2002, p. 411), Woeginger (2003, p. 91; 2005, p. 9) and Miroiu (2004, p. 362) allow the set of individuals and preferences to vary.

This paper contributes to the last approach by suggesting another axiomatization of the relative majority rule. The axioms express properties of monotonicity, efficiency and continuity. Monotonicity holds that the society formed by the union of two societies with the common collective preference p has p as preference. Efficiency says that the preference of an individual joining an indifferent society, with the empty society assumed to be indifferent, determines the preference of the new society. Continuity asserts that the addition of an individual to a society having a strict collective preference cannot reverse the collective preference.

A weakening of efficiency is shown to lead to rules obtained from the relative majority rule by adopting some tie-breaking rule. Of particular interest is the rule in which indifferences are always resolved in the same sense, because those rules can be interpreted as rules protecting the status quo represented by one of the alternatives.

Finally, both the absolute majority rule and the previous rules based on breaking ties are characterized following a common approach: a domain where the relative majority rule can be applied is identified and next the value of each preference profile outside that domain is linked with the value of some preference on it.

2. Main definitions and assumptions

Members of the set \mathbb{N} of natural numbers are names for individuals. A society is a finite non-empty subset of \mathbb{N} . The set of alternatives is $\{\alpha, \beta\}$, with $\alpha \neq \beta$. A preference over $\{\alpha, \beta\}$ is represented by a number from the set $\{-1, 0, 1\}$. If the number is 1, α is preferred to β ; if -1 , β is preferred to α ; if 0, α is indifferent to β . A preference profile for society I is a function $x_I: I \rightarrow \{-1, 0, 1\}$ assigning a preference over $\{\alpha, \beta\}$ to each member of I . For preference profile x_I and society $J \subseteq I$, x_J is the restriction of x_I to J ; that is, x_J satisfies, for all $i \in J$, $x_J(i) = x_I(i)$. Similarly, for preference profiles x_I and x_J of disjoint societies I and J , $x_{I \cup J}$ is the preference profile such that, for $K \in \{I, J\}$, x_K is the restriction of $x_{I \cup J}$ to x_K . For preference profile x_I and $i \in I$, x_i will abbreviate $x_I(i)$. For $n \in \mathbb{N}$, X_n is the set of all preference profiles x_I such that I has n elements. The set X is the set of all preference profiles x_I such that I is a society.

Definition 2.1. A social welfare function is a mapping $f: X \rightarrow \{-1, 0, 1\}$.

A social welfare function takes as input the preferences over $\{\alpha, \beta\}$ of all the members of any given society I of individuals and outputs a collective preference over $\{\alpha, \beta\}$. Thus, for $x_I \in X$: (i) $f(x_I) = 1$ means that, according to f , society I prefers α to β ; (ii) $f(x_I) = -1$, that society I prefers β to α ; and (iii) $f(x_I) = 0$, that society I is indifferent between α and β . For $x_I \in X$ and $a \in \{-1, 0, 1\}$, $n_a(x_I)$ is the number of members of the set $\{i \in I: x_i = a\}$.

Definition 2.2. The relative majority rule is the social welfare function μ such that, for all $x_I \in X$: (i) if $n_1(x_I) > n_{-1}(x_I)$, then $\mu(x_I) = 1$; (ii) if $n_1(x_I) < n_{-1}(x_I)$, then $\mu(x_I) = -1$; and (iii) if $n_1(x_I) = n_{-1}(x_I)$, then $\mu(x_I) = 0$.

Definition 2.3. The relative majority rule favouring $a \in \{1, -1\}$ is the social welfare function μ_a such that, for all $x_I \in X$: (i) if $n_a(x_I) + n_0(x_I) > n_{-a}(x_I)$, then $\mu_a(x_I) = a$; (ii) if $n_a(x_I) + n_0(x_I) < n_{-a}(x_I)$, then $\mu_a(x_I) = -a$; and (iii) if $n_a(x_I) + n_0(x_I) = n_{-a}(x_I)$, then $\mu_a(x_I) = 0$.

Definition 2.4. The absolute majority rule is the social welfare function μ^* such that, for all $x_I \in X$: (i) if $n_1(x_I) > n_{-1}(x_I) + n_0(x_I)$, then $\mu^*(x_I) = 1$; (ii) if $n_{-1}(x_I) > n_1(x_I) + n_0(x_I)$, then $\mu^*(x_I) = -1$; and (iii) otherwise, $\mu^*(x_I) = 0$.

The relative majority rule regards indifference as irrelevant. Therefore, the preference associated with x_I is obtained by cancelling out all the components that are equal to 0.

The absolute majority rule aligns indifference with the weaker side: indifference is counted as a preference for any alternative with smallest support. The relative majority rule favouring a always aligns indifference with a . This suggests that the rule embodies a preference for the status quo represented by a , since indifference is never resolved against the status quo. To illustrate the differences between μ , μ_a and μ^* , let, with some abuse of notation, $x_I = (1, -1, -1, 0, 0)$. Then $\mu(x_I) = -1$, $\mu_1(x_I) = 1$ and $\mu^*(x_I) = 0$.

MON. Monotonicity

For all $x_I \in X$ and $x_J \in X$ such that $I \cap J = \emptyset$, if $f(x_I) = f(x_J)$, then $f(x_I) = f(x_{I \cup J})$.

MON holds that if the collective preference of two disjoint societies is the same, then that preference is also the collective preference of the union of the two societies. MON can then be viewed as a form of unanimity with respect to societies. Xu and Zhong (2009) consider the weaker monotonicity requirement according to which it is only strict preference that is preserved by joining societies together.

UNA. Unanimity

For all $x_I \in X$ and $a \in \{-1, 0, 1\}$, if, for all $i \in I$, $x_i = a$, then $f(x_I) = a$.

UNA₀. Unanimity except for indifference

For all $x_I \in X$ and $a \in \{-1, 1\}$, if, for all $i \in I$, $x_i = a$, then $f(x_I) = a$.

UNA demands a unanimously held preference to be respected: if all the members of a society have the same preference, then that preference defines the collective preference. UNA₀ just demands respect for unanimous strict preference.

EFF. Efficiency

For all $x_I \in X$ and $i \in I$, if $f(x_{I \setminus \{i\}}) = 0$ or $I \setminus \{i\} = \emptyset$, then $f(x_I) = x_i$.

WEF. Weak efficiency

For all $x_I \in X$ and $i \in I$, if $f(x_{I \setminus \{i\}}) = 0$ and $f(x_i) \neq 0$, then $f(x_I) = f(x_i)$.

For the case in which I is a singleton, EFF is just UNA: the collective preference of a society with only one member is the preference of that member. When I has at least two members, EFF requires indifference to be cancellable: if an individual joins an indifferent society, then his preference determines the preference of the collective.

WEF generalizes EFF in two directions. First, if i joins an indifferent society $\Lambda\{i\}$, then the preference of I is not i 's preference but the preference that the social welfare function associates with $\{i\}$. And second, for i to determine the preference of I , $\{i\}$ must not be indifferent. WEF generalizes the property that $f(x_{\Lambda\{i\}}) = 0$ and $x_i \neq 0$ imply $f(x_I) = x_i$, according to which, for individual i joining an indifferent society to determine the collective preference, i 's preference must be resolute (strict). EFF can be viewed as a weakening of Xu and Zhong's (2009) independence of an unconcerned coalition, which establishes that if $f(x_{\Lambda\{i\}}) = 0$, then $f(x_I) = f(x_J)$.

For society $I = \{i, j\}$, $a \in \{-1, 0, 1\}$ and $b \in \{-1, 0, 1\}$, (a^i, b^j) is the member x_I of X_2 such that $x_i = a$ and $x_j = b$. Similarly, (x_I, a^i) stands for the preference profile $y_{I \cup \{i\}}$ such that $y_I = x_I$ and $y_i = a$.

CON. *Continuity*

For all $x_I \in X$ and $i \in I$, if $f(x_I) \neq 0$, then $f(x_{\Lambda\{i\}}) \neq -f(x_I)$.

CON is related to Fishburn's (1973, p. 58) weak non-reversibility, which states that $f(x_I, 1^i) = 1$ implies $f(x_I, 0^i) \neq -1$. CON requires that the addition of a single individual to a society cannot completely reverse the collective preference. Accordingly, $f(x_I) = 1$ cannot imply $f(x_I, a^i) = -1$ and $f(x_I) = -1$ cannot imply $f(x_I, a^i) = 1$. It is worth noticing that the proofs of all the results in which CON is postulated just make use of CON for the particular case in which I has two members.

SUB. *Substitutability*

For all $i \in \mathbb{N}$, $j \in \mathbb{N} \setminus \{i\}$, $k \in \mathbb{N} \setminus \{i, j\}$, $a \in \{-1, 0, 1\}$, and $b \in \{-1, 0, 1\}$, $f(a^i, b^j) = f(a^i, b^k)$.

SUB asserts that, for societies with two members, the collective preference exclusively depends on the preferences, so the identity of the individuals having those preferences is irrelevant.

3. A characterization of the relative majority rule

Remark 3.1. The relative majority rule satisfies MON, EFF and CON.

It should not be difficult to verify that the relative majority rule satisfies MON and EFF. With respect to CON, suppose $\mu(x_I) = a \neq 0$. Let $i \in I$. If $x_i \in \{0, -a\}$, then $\mu(x_{\Lambda\{i\}}) = a \neq -\mu(x_I)$. And if $x_i = a$, then $\mu(x_{\Lambda\{i\}}) \in \{0, a\}$, so again $\mu(x_{\Lambda\{i\}}) \neq -\mu(x_I)$.

Lemma 3.2. With $k \geq 2$, let f be a social welfare function such that (1) holds. Then $f = \mu$ if and only if f satisfies MON and EFF.

$$f = \mu \text{ on } X_1 \cup \dots \cup X_k \quad (1)$$

Proof. “ \Rightarrow ” Remark 3.1. “ \Leftarrow ” Taking (1) as the base case of an induction argument, choose $n > k$ and suppose that $f = \mu$ on $X_1 \cup \dots \cup X_{n-1}$. To show that $f = \mu$ on X_n , choose $x_I \in X_n$. Case 1: $\mu(x_I) = 0$. Case 1a: for some $i \in I$, $x_i = 0$. By definition of μ , $\mu(x_{\Lambda\{i\}}) = 0$. By (1), $f(x_{\Lambda\{i\}}) = \mu(x_{\Lambda\{i\}}) = 0$ and $f(x_i) = \mu(x_i) = 0$. By MON, $f(x_i) = f(x_{\Lambda\{i\}})$ implies $f(x_I) = f(x_{\Lambda\{i\}}) = 0$. Case 1b: for all $i \in I$, $x_i \neq 0$. By definition of μ , there are $i \in I$ and $j \in \Lambda\{i\}$ such that $x_i = 1$, $x_j = -1$, $\mu(x_{\{i,j\}}) = 0$, and $\mu(x_{\Lambda\{i,j\}}) = 0$. By (1), $f(x_{\Lambda\{i,j\}}) = \mu(x_{\Lambda\{i,j\}}) = 0$ and $f(x_{\{i,j\}}) = \mu(x_{\{i,j\}}) = 0$. By MON, $f(x_{\Lambda\{i,j\}}) = f(x_{\{i,j\}})$ implies $f(x_I) = f(x_{\{i,j\}}) = 0$. Case 2: $\mu(x_I) = a \in \{1, -1\}$. By definition of μ , there $i \in I$ such that $x_i = a$ and $\mu(x_{\Lambda\{i\}}) \in \{0, a\}$. Case 2a: $\mu(x_{\Lambda\{i\}}) = a$. By (1), $f(x_{\Lambda\{i\}}) = \mu(x_{\Lambda\{i\}}) = a$ and $f(x_i) = \mu(x_i) = a$. By MON, $f(x_{\Lambda\{i\}}) = f(x_i)$ implies $f(x_I) = f(x_i) = a$. Case 2b: $\mu(x_{\Lambda\{i\}}) = 0$. By (1), $f(x_{\Lambda\{i\}}) = \mu(x_{\Lambda\{i\}}) = 0$. By EFF, $f(x_{\Lambda\{i\}}) = 0$ implies $f(x_I) = x_i = a$. ■

By Lemma 3.2, for any $k \geq 2$, if all societies with size not greater than k adopt the relative majority rule, then, in order to extend the rule to all the societies, MON and EFF constitute necessary and sufficient conditions. In view of this, MON and EFF together with axioms helping to characterizing the relative majority rule for societies with at most two members will provide an axiomatization of the relative majority rule. Lemma 3.3 shows that CON is help enough.

Lemma 3.3. Let f be a social welfare function satisfying MON, EFF and CON. Then (1) holds for $k = 2$.

Proof. It must be shown that, for all $x_I \in X_1 \cup X_2$, $f(x_I) = \mu(x_I)$. Case 1: $x_I \in X_1$. Let $I = \{i\}$. By EFF, $f(x_i) = x_i = \mu(x_i)$. Case 2: $x_I \in X_2$. Let $I = \{i, j\}$. By case 1 and MON, for all $a \in \{-1, 0, 1\}$, $f(a^i, a^j) = a = \mu(a^i, a^j)$. By case 1, $f(0^i) = f(0^j) = 0$. In view of this, by EFF, for all $a \in \{-1, 1\}$, $f(0^i, a^j) = a = \mu(0^i, a^j)$ and $f(a^i, 0^j) = a = \mu(a^i, 0^j)$. Finally, consider $f(1^i, -1^j)$ and $f(-1^i, 1^j)$. Let $a \in \{-1, 1\}$. To show that $f(a^i, -a^j) = 0$, suppose not: $f(a^i, -a^j) \neq 0$. By CON, $f(a^i) \neq -f(a^i, -a^j)$ and $f(-a^j) \neq -f(a^i, -a^j)$. By case 1, $f(a^i) = a$ and $f(-a^j) = -a$. Consequently, $a \neq -f(a^i, -a^j) \neq -a$, which contradicts $f(a^i, -a^j) \neq 0$. ■

Proposition 3.4. A social welfare function f satisfies MON, EFF and CON if and only if f is the relative majority rule.

Proof. Remark 3.1, Lemma 3.2 and Lemma 3.3. ■

Remark 3.5. MON, EFF and CON also characterize the relative majority rule when defined only for the set of preference profiles in which no individual is indifferent.

4. The relative majority rule with a rule to resolve indifferences

This section considers the implications of weakening EFF into WEF in Proposition 3.4. It is shown that the resulting social welfare functions arise from the relative majority rule by resolving (maybe only some) indifferences. It is nonetheless worth noticing that the absolute majority rule does not satisfy WEF: whereas $\mu^*(1, 0, 0) = 0$, $\mu^*(1^i, 1, 0, 0) \neq \mu^*(1^i) = 1$. Section 5 will suggest another approach to make evident a connection between the absolute majority rule and the rules obtained in this section.

Definition 4.1. A rule to (possibly) resolve indifferences is a mapping $r : \mathbb{N} \times X \rightarrow \{-1, 0, 1\}$.

The interpretation of $r(i, x_I)$ is as follows. First, if $i \notin I$, then $r(i, x_I)$ is irrelevant: the pair (i, x_I) is not excluded from the domain of r in order to keep the definition of r simple. Second, if $i \in I$ and $x_i \neq 0$, then $r(i, x_I)$ is also irrelevant. Finally, if $i \in I$ and $x_i = 0$, then $r(i, x_I)$ indicates how individual i would like to break his indifference if asked to. Specifically, $r(i, x_I) = 1$ means that, if pressed to resolve his indifference, i would prefer α to β ; $r(i, x_I) = -1$, that i would prefer β to α ; and $r(i, x_I) = 0$ that, even if pressed, i would remain undecided between α and β . Rules to resolve indifferences can be motivated in a lexicographic context. An individual i indifferent between α and β is, first of all, indifferent between α and β . Despite this, if i were asked to make a choice between α and β , $r(i, x_I) \neq 0$ implies that i would choose one of the two, whereas $r(i, x_I) = 0$ implies that i would remain unable to discriminate between α and β .

Let r be a rule to resolve indifferences. For $x_I \in X$, define x_I^r to be the member of X such that, for all $i \in I$: (i) if $x_i \neq 0$, then $x_i^r = x_i$; and (ii) if $x_i = 0$, then $x_i^r = r(i, x_I)$. The preference profile x_I^r is obtained from x_I by just resolving indifferences following r .

Given a rule r to resolve indifferences, a social welfare function f that is required to take r into account will have to be redefined in order to make f associate with x_I the value $f(x_I')$ instead of the value $f(x_I)$. The following definition is the result of requiring the relative majority rule to resolve indifferences following any given rule r .

Definition 4.2. The relative majority rule based on the rule r to resolve indifferences is the social welfare function $\mu^r : X \rightarrow \{-1, 0, 1\}$ such that, for all $x_I \in X$, $\mu^r(x_I) = \mu(x_I')$.

Suppose that all the individuals resolve indifferences always in the same way: there is $a \in \{-1, 0, 1\}$ such that, for all $(i, x_I) \in \mathbb{N} \times X$, $r(i, x_I) = a$. If $a = 0$, then the resulting social welfare function is the relative majority rule. And if $a \neq 0$, it is the relative majority rule favouring a .

Definition 4.3. A rule r to resolve indifferences is independent of preferences (or, for short, an independent rule) if, for all $i \in \mathbb{N}$, $x_I \in X$ and $y_I \in X$, $r(i, x_I) = r(i, y_I)$.

An independent rule can be expressed just in the form $r : \mathbb{N} \rightarrow \{-1, 0, 1\}$, where $r(i)$ indicates how i would like his indifference to be resolved. The presumption that an individual determines his own preference without taking into account the preferences of the rest of individuals justifies that his decision of how to (possibly) resolve his indifference should also be based exclusively on personal (or, more precisely, non-strategic) considerations. If this criterion is adopted, the rule is independent of preferences.

Remark 4.4. For each independent rule r , μ^r satisfies MON, WEF, CON, and UNA_0 .

Remark 4.4 follows from the fact that $\mu^r(x_I) = \mu(x_I')$ and the fact that μ satisfies MON, WEF, CON, and UNA_0 . Lemma 4.5 next is a generalization of Lemma 3.2.

Lemma 4.5. With $k \geq 2$ and $a \in \{1, -1\}$, let f be a social welfare function such that (2) holds for some independent rule r . Then $f = \mu^r$ if and only if f satisfies MON and WEF.

$$f = \mu^r \text{ on } X_1 \cup \dots \cup X_k \quad (2)$$

Proof. “ \Rightarrow ” Remark 4.4. “ \Leftarrow ” Taking (2) as the base case of an induction argument, choose $n > k$ and suppose that $f = \mu^r$ on $X_1 \cup \dots \cup X_{n-1}$. To show that $f = \mu^r$ on X_n , choose $x_I \in X_n$.

Case 1: $\mu^r(x_I) = 0$. Therefore, $\mu(x_I^r) = 0$. Case 1a: for some $i \in I$, $x_i^r = 0$. Thus, $r(i, x_I) = 0$, so $x_i = x_i^r = 0$. By definition of μ , $\mu(x_{\Lambda\{i\}}^r) = 0$. By (2) and the definition of μ^r , $f(x_{\Lambda\{i\}}) = \mu^r(x_{\Lambda\{i\}}) = \mu(x_{\Lambda\{i\}}^r) = 0$ and $f(x_i) = \mu^r(x_i) = \mu(x_i^r) = \mu(x_i) = 0$. By MON, $f(x_I) = f(x_{\Lambda\{i\}})$ implies $f(x_I) = f(x_{\Lambda\{i\}}) = 0 = \mu^r(x_I)$. Case 1b: for all $i \in I$, $x_i^r \neq 0$. By definition of μ , $\mu(x_I^r) = 0$ implies that, for some $i \in I$ and $j \in \Lambda\{i\}$, $x_i^r = 1$, $x_j^r = -1$, $\mu(x_{\{i,j\}}^r) = 0$, and $\mu(x_{\Lambda\{i,j\}}^r) = 0$. By (2) and the definition of μ^r , $f(x_{\Lambda\{i,j\}}) = \mu^r(x_{\Lambda\{i,j\}}) = \mu(x_{\Lambda\{i,j\}}^r) = 0$ and $f(x_{\{i,j\}}) = \mu^r(x_{\{i,j\}}) = \mu(x_{\{i,j\}}^r) = 0$. By MON, $f(x_{\Lambda\{i,j\}}) = f(x_{\{i,j\}})$ implies $f(x_I) = f(x_{\{i,j\}}) = 0 = \mu^r(x_I)$.

Case 2: $\mu^r(x_I) = a \neq 0$. By definition of μ^r , $a = \mu^r(x_I) = \mu(x_I^r)$. By definition of μ , there is $i \in I$ such that $x_i^r = a$ and $\mu(x_{\Lambda\{i\}}^r) \in \{0, a\}$. Case 2a: $\mu(x_{\Lambda\{i\}}^r) = a$. By (2) and the definition of μ^r , $f(x_{\Lambda\{i\}}) = \mu^r(x_{\Lambda\{i\}}) = \mu(x_{\Lambda\{i\}}^r) = a$ and $f(x_i) = \mu^r(x_i) = \mu(x_i^r) = a$. By MON, $f(x_{\Lambda\{i\}}) = f(x_i)$ implies $f(x_I) = f(x_i) = a = \mu^r(x_I)$. Case 2b: $\mu(x_{\Lambda\{i\}}^r) = 0$. By (2) and the definition of μ^r , $f(x_{\Lambda\{i\}}) = \mu^r(x_{\Lambda\{i\}}) = \mu(x_{\Lambda\{i\}}^r) = 0$ and $f(x_i) = \mu^r(x_i) = \mu(x_i^r) = a$. By WEF, $f(x_{\Lambda\{i\}}) = 0$ and $f(x_i) \neq 0$ imply $f(x_I) = f(x_i) = a = \mu^r(x_I)$. ■

Lemma 4.6. Let f be a social welfare function satisfying MON, WEF, CON, and UNA_0 . Then, for some independent rule r , (2) holds for $k = 2$.

Proof. Define, for all $i \in \mathbb{N}$ and $x_I \in X$, $r(i, x_I) = f(0^i)$. It must be shown that, all $x_I \in X_1 \cup X_2$, $f(x_I) = \mu^r(x_I)$. That is, for all $x_I \in X_1 \cup X_2$, $f(x_I) = \mu(x_I^r)$. Case 1: $x_I \in X_1$. Let $I = \{i\}$. By UNA_0 , if $x_i \neq 0$, then $f(x_i) = x_i = \mu(x_i^r)$. If $x_i = 0$, then $f(x_i) = r(i, x_I) = \mu(x_i^r)$. Case 2: $x_I \in X_2$. Let $I = \{i, j\}$ and $a \in \{-1, 1\}$. Case 2a: $x_i = x_j = a$. By case 1, $f(x_i) = f(x_j) = a$. By MON, $f(x_I) = f(x_i) = a = \mu(x_I^r)$. Case 2b: $x_i = a$ and $x_j = -a$. Suppose $f(x_I) \neq 0$. By CON, $f(x_i) \neq -f(x_I)$ and $f(x_j) \neq -f(x_I)$. By case 1, $f(x_i) = a$ and $f(x_j) = -a$. As a result, $a \neq -f(x_I) \neq -a$, which contradicts $f(x_I) \neq 0$. Therefore, $f(x_I) = 0 = \mu(x_I^r)$. Case 2c: $0 \in \{x_i, x_j\}$. Without loss of generality, assume $x_i = 0$. Case 2c1: $f(0^i) = 0$. If $f(x_j) \neq 0$, then, by WEF, $f(x_I) = f(x_j) = \mu^r(x_j) = \mu(x_j^r) = \mu(x_I^r)$. If $f(x_j) = 0$, then, by case 1, $x_j = 0$. By MON, $f(0^i) = f(0^j)$ implies $f(x_I) = f(x_j) = 0 = \mu(x_I^r)$. Case 2c2: $f(0^i) = a \neq 0$. If $x_j = a$, by case 2a, $f(x_I) = \mu(x_I^r)$. If $x_j = -a$, by case 2b, $f(x_I) = \mu(x_I^r)$. If $x_j = 0$, by permuting i and j in case 2c1, $f(x_I) = \mu(x_I^r)$. ■

Proposition 4.7. A social welfare function satisfies MON, WEF, CON, and UNA_0 if and only if f is the relative majority rule based on some independent rule r .

Proof. Remark 4.4, Lemma 4.5 and Lemma 4.6. ■

Proposition 4.8. A social welfare function satisfies MON, WEF, CON, and UNA if and only if f is the relative majority rule.

Proof. “ \Leftarrow ” Remark 4.4. “ \Rightarrow ” By Proposition 3.4, it is enough to show that EFF holds. Let $x_I \in X$ and $i \in I$. Case 1: $I = \{i\}$. By UNA, $f(x_i) = x_i$. Case 2: $I \neq \{i\}$ and $f(x_{I \setminus \{i\}}) = 0$. If $f(x_i) \neq 0$, then, by WEF and case 1, $f(x_I) = f(x_i) = x_i$. If $f(x_i) = 0$, then, by MON, $f(x_i) = f(x_{I \setminus \{i\}})$ imply $f(x_I) = f(x_i) = x_i$, the last step by case 1. ■

Proposition 4.9. A social welfare function satisfies MON, WEF, CON, UNA₀, and SUB if and only if f is the relative majority rule or, for some $a \in \{1, -1\}$, the relative majority rule favouring a .

Proof. “ \Leftarrow ” It is not difficult to verify that μ_a satisfies SUB. Remark 4.4 does the rest of the job. “ \Rightarrow ” Choose $i \in \mathbb{N}$. If $f(0^i) = 0$, then, by SUB, for all $j \in \mathbb{N}$, $f(0^j) = 0$. By Proposition 4.7, f must be the relative majority rule. If $f(0^i) = a \neq 0$, then, by SUB, for all $j \in \mathbb{N}$, $f(0^j) = a$. By Proposition 4.7, f must be the relative majority rule favouring a . ■

5. The relative majority rule on restricted domains

For a rule r to resolve indifferences, let $X^r = \{x_I \in X: x_I^r = x_I\}$. The members of X^r are those preference profiles that r leaves unaffected. Notice that X^r satisfies the following: (i) if $x_I \in X^r$ and $x_J \in X^r$, then $x_{I \cup J} \in X^r$; and (ii) if $x_I \in X^r$ and $i \in I$, then $x_{I \setminus \{i\}} \in X^r$. Notice as well that, for all $x_I \in X^r$, $\mu^r(x_I) = \mu(x_I)$. Hence, X^r defines a core of minimum agreement between the relative majority rule μ and μ^r . This observation suggests another way of characterizing μ^r by restricting to X^r the domain of applicability of axioms MON, EFF and CON that have characterized μ in Proposition 3.4. In this respect, for $A \in \{\text{MON}, \text{EFF}, \text{CON}\}$, let A^r be the axiom obtained from A by replacing X with X^r . The following axiom captures the idea that the social welfare function turns to itself when it is required to follow a tie-breaking rule r .

PAI^r. *Pairing*

For all $x_I \in X \setminus X^r$, $f(x_I) = f(x_I^r)$.

Proposition 5.1. Let r be a rule to resolve indifferences. A social welfare function f satisfies MON^r, EFF^r, CON^r, and PAI^r if and only if f is the relative majority rule based on r .

Proof. “ \Leftarrow ” Remark 4.4 and the fact that, for all $x_I \in X \setminus X^r$, $\mu^r(x_I) = \mu(x_I^r) = \mu((x_I^r)^r) = \mu^r(x_I^r)$. “ \Rightarrow ” The proofs of Lemmas 3.2 and 3.3 for domain X remain valid for domain X^r , so $f = \mu$ on X^r . Since $\mu = \mu^r$ on X^r , $f = \mu^r$ on X^r . On the other hand, for all $x_I \in X \setminus X^r$,

by PAI^r, $f(x_I) = f(x_I^r)$. Clearly, $x_I^r \in X^r$. By the agreement of f with μ^r on X^r , $f(x_I^r) = \mu^r(x_I^r)$. By definition of μ^r , $\mu^r(x_I^r) = \mu((x_I^r)^r) = \mu(x_I^r) = \mu^r(x_I)$. In short, $f = \mu^r$ on $X \setminus X^r$. ■

By Proposition 5.1, a relative majority rule based on r can be seen as the result of a two-stage process. First, the domain X^r is identified, which constitutes a safe domain for the relative majority rule to be applied. And second, for every preference profile x_I outside that domain, a partner is found in X^r from which x_I takes its value. It is next shown that a similar logic can be used to characterize the absolute majority rule.

For a given social welfare function f , define $X' = \{x_I \in X: \text{for every rule } r \text{ to resolve indifferences, } f(x_I) = f(x_I^r)\}$. Notice that X' satisfies the following: (i) if $x_I \in X'$ and $x_J \in X'$, then $x_{I \cup J} \in X'$; and (ii) if $x_I \in X'$ and $i \in I$, then $x_{I \setminus \{i\}} \in X'$. For $A \in \{\text{MON}, \text{EFF}, \text{CON}\}$, let A' be the axiom obtained from A by replacing X with X' . An interesting difference between axiom A and A' is that the domain on which A' acts is endogenous, in the sense that it depends on the f for which A' is postulated.

DIS. *Disagreement*

For all $x_I \in X \setminus X'$, $f(x_I) = 0$.

Consider social welfare function f . Suppose that, for this f , not all the rules to resolve indifferences produce the same result. That is, the collective preference will in general depend on the rule chosen. Given this dependence on an arbitrary choice, it may appear reasonable to opt for indifference as the collective preference. Otherwise, there could be an opportunity to manipulate the collective preference by changing the tie-breaking rule. This is precisely what DIS demands: if the result depends on how indifferences can be resolved, then declare indifference as the way to deal with this disagreement.

Proposition 5.2. A social welfare function f satisfies MON', EFF', CON', and DIS if and only if f is the absolute majority rule.

Proof. “ \Leftarrow ” Easily verified. “ \Rightarrow ” The proofs of Lemmas 3.2 and 3.3 for domain X remain valid for domain X' , so $f = \mu$ on X' . Since $\mu = \mu^*$ on X' , it follows that $f = \mu^*$ on X' . In addition, for all $x_I \in X \setminus X'$, $\mu^*(x_I) = 0$. Therefore, by DIS, $f = \mu^*$ on X' . ■

Interestingly, the absolute majority rule is the only qualified majority rule that can be justified in terms of the relative majority rule and the adoption of rules to resolve indifferences. For instance, if f is any qualified majority such that $f(x_I) = 0$, where $x_I =$

(1, 1, 1, 1, -1, -1), then, by changing indifferences, x_i cannot be associated with another preference profile that yields 0 under the relative majority rule.

The results in Llamazares (2006, pp. 318–319) show that a property of cancellability links the relative majority and the qualified majorities; see Houy (2006, 2007) for additional characterizations of qualified majorities. In the present setting, cancellability amounts to replacing two opposite preferences (1, -1) by an indifference. This suggests that the link is established through the removal of individuals. The results in this paper point to the resolution of indifference as another link, which is weaker than the removal of individuals, because the connection between the relative majority and the rest of qualified majorities just stops at the first step: the absolute majority.

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