Stable consistent allocation

Antonio Quesada[†]

Departament d'Economia, Universitat Rovira i Virgili, Avinguda de la Universitat 1, 43204 Reus, Spain

14th December 2008

156.1

Abstract

The problem of allocating indivisible goods is considered when a measure of power is exogenously defined for individuals and coalitions. If some individual is the weakest, allocation rules having a hierarchy of dictators are the only rules satisfying stability, consistency and Pareto efficiency for allocation problems with one individual. Stability means that there is some coalition structure under which individuals have no incentive to form another coalition and use the power of that coalition to obtain an allocation in which all the members of the coalition are better off. Consistency says that if some individual i departs with the assigned good, the allocation mechanism does not modify the goods assigned to the rest of individuals in the allocation problem induced by i's departure.

Keywords: Allocation rule, hierarchy of dictators, indivisible good, strict preference, power, coalition formation.

JEL Classification: D61, D70, D85, C72

[†] E-mail address: aqa@urv.cat. Financial support from the Spanish *Ministerio de Educación y Ciencia*, under research project SEJ2007-67580-C02-01, and from the *Departament d'Universitats, Recerca i Societat de la Informació* (*Generalitat de Catalunya*), under research project 2005SGR-00949, is gratefully acknowledged.

1. Introduction

Allocation problems are typically analyzed under the presumption that exchange is voluntary and that any allocation solving the problem is enforceable. But in many real allocation problems it may occur that no law regulates exchanges, that the regulating laws are not enforceable or that those infringing these laws have means to avoid prosecution or punishment. In such cases, allocation is, in the last instance, dictated by force. This paper considers the problem of allocating indivisible goods when one cannot prevent agents from using force to obtain what they want.

The analysis of this problem is carried out in a simplified version of the house allocation model by Shapley and Scarf (1974). The simplifications are two: preferences over the goods are assumed to be strict and agents do not initially own the goods as endowments. The results already obtained in this more simple model indicate that some hierarchical structure among individuals determines the allocation of the goods; see, for instance, Svensson (1999), Ergin (2000), Pápai (2000a), Ehlers (2002) or Ehlers and Klaus (2003). Analogous results have been obtained in more general versions of the model involving quotas (Svensson (1994), Ehlers and Klaus (2006)), property rights over several goods (Pápai (2003)), indifferences (Bogomolnaia et al. (2005)) or preferences over sets of goods (Pápai (2000b, 2001)).

In all those contributions, the axioms imposed on the allocation mechanism turn out to implicitly attribute "power" to individuals. This paper models this "power" explicitly: some power function measures the power of each individual. The power of a coalition of individuals is defined as the sum of the power of the individuals in the coalition (the results in the paper hold if one assumes instead that coalitional power is superadditive). Coalitions can be formed for two reasons: to protect the goods allocated to the members of the coalition or to take the goods allocated to the members of a weaker coalition. The first condition imposed on an allocation mechanism is that it should be stable with respect to some coalition structure. Loosely speaking, this means that, for some partition of the set of individuals into coalitions, no coalition can be formed to take the goods assigned to a weaker coalition and distribute them into the members of the attacking coalition in such a way that everybody is better off. This requirement is weak because it just demands stability with respect to least one coalition structure: some network of alliances prevents power to interfere in the allocation.

Two additional axioms are postulated. One is consistency, a property invoked in game theory to solve strategic problems; see, for instance, Thomson (1990). In the present context, consistency means that if an individual departs with the assigned good then, in the induced allocation problem, each individual retains the good assigned before the individual departed. The other axiom is a weak form of Pareto efficiency: for allocation problems involving only one individual, the individual must be assigned his most preferred good.

The results obtained in this paper are strongly related to those by Ergin (2000), who also imposes consistency properties on the allocation mechanism. He shows (Theorem 1) that just pairwise consistency and pairwise neutrality induce some partial hierarchy of dictators when the allocation problem involves the same number of individuals and goods. Proposition 3.2 (which postulates stability and consistency) is a similar result, with the difference that the hierarchy is partial in only excluding two individuals (who, in any case, lie at the bottom of the hierarchical structure). A complete hierarchy is obtained in Proposition 3.4 by requiring some individual to be the weakest in power. In contrast to Ergin's (2000, p. 86) Corollary 1, the complete hierarchy arises without postulating any Pareto efficiency principle.

Propositions 3.5 and 3.6 extend Proposition 3.2 to the general case, no covered by Ergin (2000), in which the number of individuals and goods may differ. With the addition of the above weak Pareto principle, stability and consistency again imply the existence of a partial hierarchy that only excludes two individuals. The main result of the paper, Proposition 3.7, states that the complete hierarchy of dictators is characterized by the weak Pareto efficiency condition, consistency and the version of stability in which some individual is the weakest. The proof of Proposition 3.7 shows that the individuals higher the allocation hierarchy are those higher in the power hierarchy. This suggests that consistency and Pareto efficiency can be viewed as properties transforming exogenous power into economic power, once a basic notion of stability with respect to the use of the exogenous power is postulated.

2. Definitions and assumptions

Let $N = \{1, \ldots, n\}$ be the set consisting of the first $n \geq 1$ positive integers. Members of N are names for individuals. Let X be a finite set whose $m \geq 1$ members represent objects (or tasks or anything that can be assigned to individuals). With $\varnothing$ standing for the absence of object, for non-empty $Y \subseteq X$, L_Y is the set of strict preferences (linear orderings) that can be defined on $Y \cup \{\varnothing\}$ such that, for all $x \in Y$, x is preferred to $\varnothing$. This presumes that all members of Y are desirable. As usual, for $p \in L_Y$, $x \in Y$ and $y \in$

$Y\backslash\{x\}$, $x\ p\ y$ means that x is preferred to y. The set D is the set of all functions $P_{IY} : I \to L_Y$ such that $I \subseteq N$, $Y \subseteq X$ and $I \neq \varnothing \neq Y$. A member of D represents a way of assigning a preference defined on $Y \cup \{\varnothing\}$ to every individual in the group I. For $I \subseteq N$, $Y \subseteq X$, $J \subseteq N$ and $Z \subseteq X$, $P_{IY}|_{JZ}$ is the member Q_{JZ} of D such that, for all $i \in J$, $Q_{JZ}(i)$ is $P_{IY}(i)$ restricted to Z; that is, for all $i \in J$, $x \in Z$ and $y \in Z\backslash\{x\}$, $x\ Q_{JZ}(i)\ y$ if, and only if, $x\ P_{IY}(i)\ y$. For a finite set S, $|S|$ designates the number of members of S.

Definition 2.1. For $I \subseteq N$ and $Y \subseteq X$ with $I \neq \varnothing \neq Y$, an allocation is a mapping $\alpha_{IY} : I \to Y \cup \{\varnothing\}$ such that $|\{x \in Y\text{: for some } i \in I, \alpha_{IY}(i) \neq \varnothing\}| = min\{|Y|, |I|\}$.

Given $I \subseteq N$ and $Y \subseteq X$, an allocation is a way of assigning objects in Y to individuals in I so that: (i) no object is assigned to two different individuals; (ii) if $|Y| \leq |I|$ then every object is assigned to some individual; and (iii) if $|Y| > |I|$ then every individual is assigned some object. For $I \subseteq N$ and $Y \subseteq X$ with $I \neq \varnothing \neq Y$, A_{IY} is the set of all the allocations α_{IY} and A is the union, over all pairs (I, Y) such that $I \subseteq N$ and $Y \subseteq X$ with $I \neq \varnothing \neq Y$, of the sets A_{IY}.

Definition 2.2. For $D' \subseteq D$, an allocation rule (on D') is a mapping $f : D' \to A$ such that, for all $P_{IY} \in D'$, $f(P_{IY}) \in A_{IY}$.

Any member P_{IY} of $D' \subseteq D$ expresses the allocation problem in which (possibly, only some of) the objects in Y must be distributed among (possibly, only some of) the individuals in I. An allocation rule on $D' \subseteq D$ is a way of solving each such allocation problem. For $i \in I$, $f_i(P_{IY})$ will designate the member of $Y \cup \{\varnothing\}$ assigned to i in $f(P_{IY})$.

Definition 2.3. For $k \leq n$ and $D' \subseteq D$, allocation rule $f : D' \to A$ has a hierarchy$_k$ of dictators if there is a sequence $(i_1, \ldots, i_k)$ of k different members of N satisfying, for all $P_{IY} \in D'$, the following. Set $r = min\{|I|, |Y|, k\}$ and let $(j_1, \ldots, j_r)$ consists of the first r members of $(i_1, \ldots, i_k)$ that belong to I. Then: (i) $f_{j_1}(P_{IY})$ is j_1's most preferred object in Y according to $P_{IY}(j_1)$; and (ii) for $t \in \{2, \ldots, r\}$, $f_{j_t}(P_{IY})$ is j_t's most preferred object in $Y\backslash\{f_{j_1}(P_{IY}), \ldots, f_{j_{t-1}}(P_{IY})\}$ according to $P_{IY}(j_t)$. An allocation rule has a hierarchy of dictators if it has a hierarchy$_n$ of dictators.

When an allocation rule has a hierarchy$_k$ of dictators, there is a ranking of k individuals such that, for each allocation problem solved by the rule, every individual i in the ranking is assigned his most preferred object among those left, if any, by the individuals involved in the allocation problem and coming before i in the ranking.

Definition 2.4. A power function is a mapping $w : N \to \mathbb{N}\backslash\{0\}$ that assigns a positive integer $w(i)$ to each individual i.

Any power function w is assumed to be extended to the set of subsets of N as follows: for all $I = \{i_1, \dots , i_k\} \subseteq N$ with $k \geq 2$, $w(I) = w(i_1) + \dots + w(i_k)$. This says that the power of a coalition is the sum of the power of the members of the coalition. All the results in the paper hold if the extension is assumed to be weakly superadditive ($w(I) \geq w(i_1) + \dots + w(i_k)$) or superadditive ($w(I) > w(i_1) + \dots + w(i_k)$).

Definition 2.5. A power* function is a power function $w : N \to \mathbb{N}\backslash\{0\}$ such that, for some $i \in N$ and all $j \in N\backslash\{i\}$, $w(i) < w(j)$.

A partition of a finite set S is a set $\{S_1, \dots , S_k\}$ such that: (i) $S = S_1 \cup \dots \cup S_k$; (ii) for all $r \in \{1, \dots , k\}$ and $t \in \{1, \dots , k\}\backslash\{r\}$, $S_r \cap S_t = \emptyset$; and (iii) for all $r \in \{1, \dots , k\}$, $S_r \neq \emptyset$. For $I \subseteq N$, a coalition structure C_I on I is a partition of the set I.

Definition 2.6. For $H \subseteq N$ and $Y \subseteq X$ with $H \neq \emptyset \neq Y$, allocation α_{HY} is stable with respect to power function w, coalition structure C_H and preference profile P_{HY} if there do not exist $K \in C_H$, $I \subset K$, $J \subseteq H\backslash K$ and allocation $\beta_{HY} \in A_{HY}$ such that: (i) $w(I \cup J) > w(K\backslash I)$; (ii) for all $i \in I \cup J$, $\beta_{HY}(i)\ P_{HY}(i)\ \alpha(i)$; and (iii) for all $i \in H\backslash(K \cup J)$, $\beta_{HY}(i) = \alpha_{HY}(i)$.

An allocation α_{HY} is stable with respect to power function w, coalition structure C_H and preference profile P_{HY} if no coalition K in the structure C_H can be plundered. This means that no subset I of K can join forces with members J outside K to create a coalition $I \cup J$ having more power, according to w, than the residual coalition $K\backslash I$ and use that power to reallocate the objects that α_{HY} assigns to the members of $K \cup J$ in such a way that all the members of $I \cup J$ are better off, according to the preferences in P_{HY}.

Four properties of an allocation rule f are suggested next: the stability requirements STA and STA*; the consistency principle CON; and the Paretian condition PAR_1.

STA. There is a power function w satisfying the following: for every $P_{IY} \in D$, there is some coalition structure C_I such that $f(P_{IY})$ is stable with respect to w, C_I and P_{IY}.

STA is a property of allocation stability: some power function w is such that, for every allocation $f(P_{IY})$, some coalition structure on I ensures the stability of the allocation.

STA*. There is a power* function w satisfying the following: for every $P_{IY} \in D$, there is some coalition structure C_I such that $f(P_{IY})$ is stable with respect to w, C_I and P_{IY}.

STA* differs from STA in requiring stability to arise from a power* function, that is, a power function in which a unique individual has the smallest amount of individual power.

CON. For all $I \subseteq N$, $Y \subseteq X$, $P_{IY} \in D$, $i \in I$ and $j \in I \backslash \{i\}$, if $J = I \backslash \{j\}$ and $Z = X \backslash \{f_j(P_{IY})\}$ then $f_i(P_{IY}|_{JZ}) = f_i(P_{IY})$.

CON expresses the idea that the allocation is consistent in the sense that if one individual leaves the allocation problem together with the object he receives then, in the induced allocation problem, the individuals are assigned the same objects that were assigned in the initial problem. As a result, the solution of the original allocation problem is projected onto the reduced problem. CON can also be viewed as a robustness property: when one individual departs with the assigned object, all the other objects remain in the same hands.

PAR$_1$. For all $P_{IY} \in D$, if $I = \{i\}$ then $f_i(P_{IY})$ is i's most preferred object in Y according to $P_{IY}(i)$.

PAR$_1$ is a weak form of Paretian efficiency: in allocation problems involving a single individual, he must be assigned his most preferred object.

3. Results

For $P_{IY} \in D$, $i \in I$ and $j \in I \backslash \{i\}$, i envies j in $f(P_{IY})$ if $f_j(P_{IY})\ P_{IY}(i)\ f_i(P_{IY})$.

Lemma 3.1. For $D' \subseteq D$, let allocation rule $f : D' \rightarrow A$ satisfy CON and STA, with power function w. Suppose there are $P_{IY} \in D$, $i \in I$, $j \in I \backslash \{i\}$ and $k \in I \backslash \{j, k\}$ such that $f_i(P_{IY}) \neq \varnothing \neq f_j(P_{IY})$, k envies both i and j in $f(P_{IY})$ and j envies i in $f(P_{IY})$. Then $w(j) < w(i) > w(k)$.

Proof. Let D', f, w, P_{IY}, i, j and k be as required. Define $J = \{i, j, k\}$, $Z = \{f_i(P_{IY}), f_j(P_{IY}), f_k(P_{IY})\}$ and $P_{JZ} = P_{IY}|_{JZ}$. By assumption, $|Z| \geq 2$. By successive application of CON, for all $r \in J$, $f_r(P_{JZ}) = f_r(P_{IY})$. Since k envies i and j in $f(P_{IY})$ and j envies i in $f(P_{IY})$, it follows that k envies i and j in $f(P_{JZ})$ and j envies i in $f(P_{JZ})$. Hence, $f_i(P_{JZ})\ P_{JZ}(k)\ f_k(P_{JZ})$,

$f_j(P_{JZ})\ P_{JZ}(k)\ f_k(P_{JZ})$ and $f_i(P_{JZ})\ P_{JZ}(j)\ f_j(P_{JZ})$. By STA, for some coalition structure C_J on J, $f(P_{JZ})$ is stable with respect to w, C_J and P_{JZ}. This implies that there is no allocation $\beta_{JZ} \in A_{JZ}$ such that: (i) $w(\{j, k\}) > w(i)$; and (ii) for all $r \in J\backslash\{i\}$, $\beta_{JZ}(r)\ P_{JZ}(r)\ f_r(P_{JZ})$. Define $\beta_{JZ} \in A_{JZ}$ to be the allocation such that $\beta_{JZ}(j) = f_i(P_{JZ})$ and $\beta_{JZ}(k) = f_j(P_{JZ})$. As β_{JZ} satisfies (ii), (i) must fail. Accordingly, $w(\{j, k\}) \leq w(i)$. By definition of power function, $w(j) > 0 < w(k)$. Hence, $w(\{j, k\}) > w(j)$ and $w(\{j, k\}) > w(k)$. As a result, for all $r \in \{j, k\}$, $w(i) > w(r)$.■

Proposition 3.2. Let $m = n \geq 3$ and $D^* = \{P_{IY} \in D: |I| = |Y|\}$. If allocation rule $f : D^* \rightarrow A$ satisfies STA and CON then f has a hierarchy$_{n-2}$ of dictators.

Proof. Let $X = \{x_1, \ldots, x_n\}$. Assume that f satisfies CON and STA, with power function w. Define $Q_{NX} \in D^*$ to be such that, for all $i \in N$ and $k \in \{1, \ldots, n\}$, x_k is the kth most preferred object of X in $Q_{NX}(i)$. For $k \in \{1, \ldots, n\}$, let $i_k \in N$ satisfy $f_{i_k}(Q_{NX}) = x_k$. By Lemma 3.1,

$$w(i_1) > w(i_2) > \ldots > w(i_{n-2}) > w(i_{n-1}) \text{ and } w(i_{n-2}) > w(i_n). \tag{1}$$

The proof amounts to showing that $(i_1, \ldots, i_{n-2})$ is a hierarchy$_{n-2}$ of dictators. To this end, choose $P_{IY} \in D^*$ and suppose that $f(P_{IY})$ does not agree with any allocation that would generate $(i_1, \ldots, i_{n-2})$ as a hierarchy$_{n-2}$. Let i_k be the first member in the sequence $(i_1, \ldots, i_{n-2})$ not assigned the object $x \in Y$ that i_k would be assigned if $(i_1, \ldots, i_{n-2})$ determined the allocation. Since $|I| = |Y|$ and $f(P_{IY})$ is an allocation of all the members of Y among the members of I, there is $i \in I\backslash\{i_1, \ldots, i_k\}$ such that $f_i(P_{IY}) = x$. Define $J = \{i_k, i\}$, $Z = \{f_{i_k}(P_{IY}), f_i(P_{IY})\}$ and $P_{JZ} = P_{IY}|_{JZ}$. By successive application of CON, $f_i(P_{JZ}) = f_i(P_{IY}) = x$. But then, by (1), $w(i_k) > w(i)$, which implies that no coalition structure C_J on J makes $f(P_{JZ})$ stable with respect to w, C_J and P_{JZ}. This contradicts STA.■

Remark 3.3. When $m = n \geq 3$, an allocation rule $f : D^* \rightarrow A$ satisfying STA and CON need not have a hierarchy$_{n-1}$ of dictators.

For instance, with $X = \{x, y, z\}$ and $N = \{1, 2, 3\}$, consider the allocation rule $f : D^* \rightarrow A$ such that: (i) 1 is always assigned his most preferred object; (ii) 2 and 3 are assigned objects respecting Pareto efficiency; and (iii) in case of conflict, 2 has priority over 3 to be assigned x and 3 has priority over 2 to be assigned y and z. Whereas f does not have a hierarchy$_2$ of dictators, it satisfies CON and STA with power function $w(1) = 3$ and $w(2) = w(3) = 1$.

Proposition 3.4. Let $m = n$ and $D^* = \{P_{IY} \in D\colon |I| = |Y|\}$. Allocation rule $f : D^* \to A$ satisfies STA* and CON if, and only if, f has a hierarchy of dictators.

Proof. "⇐" Suppose f has the hierarchy of dictators $(i_1, \dots, i_n)$. It is clear that f satisfies CON. As regards STA*, define w to be the power function such that $w(i_n) = 1$ and, for all $k \in \{1, \dots, n-1\}$, $w(i_k) = 1 + w(i_{k+1}) + \dots + w(i_n)$. This implies that, for all $k \in \{1, \dots, n-1\}$, $w(i_k) = 2w(i_{k+1})$. Hence, for all $k \in \{1, \dots, n\}$, $w(i_k) = 2^{n-k}$. Consequently, for all $P_{IY} \in D$, $f(P_{IY})$ is stable with respect to w, C_I and P_{IY}, where C_I is the coalition structure on I in which every coalition has only one member.

"⇒" There is nothing to prove if $n = 1$. Case 1: $n = 2$. Let $N = \{i, j\}$. By definition of power* function, either $w(i) > w(j)$ or $w(j) > w(i)$. Suppose $w(i) > w(j)$. It must be shown that, for all $P_{NX} \in D^*$, i is assigned his most preferred object in X according to $P_{NX}(i)$. By STA*, for some coalition structure C_N on N, $f(P_{IY})$ must stable with respect to w, C_N and P_{IY}. As $w(i) > w(j)$, an allocation not assigning i his most preferred object cannot be stable. Case 2: $n \geq 3$. Since STA* implies STA, by Proposition 3.2, f has a hierarchy$_{n-2}$ of dictators $(i_1, \dots, i_{n-2})$. Let $N \backslash \{i_1, \dots, i_{n-2}\} = \{i, j\}$. By Lemma 3.1, for all $k \in \{i_1, \dots, i_{n-2}\}$, $w(k) > w(i)$ and $w(k) > w(j)$. Given this, by definition of power* function, either $w(i) > w(j)$ or $w(j) > w(i)$. Suppose $w(i) > w(j)$. It must be then shown that f has the hierarchy of dictators $(i_1, \dots, i_{n-2}, i, j)$. By CON, it is enough to prove that, for all $P_{IY} \in D^*$ such that $I = \{i, j\}$, i is assigned his most preferred object in Y according to $P_{IY}(i)$. But this follows from $w(i) > w(j)$ and the requirement that, for some coalition structure C_I on I, $f(P_{IY})$ must stable with respect to w, C_I and P_{IY}.■

Proposition 3.5. For $m \geq n \geq 3$, if allocation rule $f : D \to A$ satisfies STA, CON and PAR$_1$ then f has a hierarchy$_{n-2}$ of dictators.

Proof. Let $X = \{x_1, \dots, x_m\}$. Assume that f satisfies PAR$_1$, CON and STA, with power function w. Define $Q_{NX} \in D^*$ to be such that, for all $i \in N$ and $k \in \{1, \dots, m\}$, x_k is the kth most preferred object of X in $Q_{NX}(i)$. For $k \in \{1, \dots, n\}$, let $i_k \in N$ satisfy $f_{i_k}(Q_{NX}) = x_k$. By Lemma 3.1, (1) holds. It must be shown that $(i_1, \dots, i_{n-2})$ is a hierarchy$_{n-2}$ of dictators. Choose $P_{IY} \in D$ and suppose that $f(P_{IY})$ does not agree with any allocation that would generate $(i_1, \dots, i_{n-2})$ as a hierarchy$_{n-2}$. Let i_k be the first member in the sequence $(i_1, \dots, i_{n-2})$ not assigned the object $x \in Y$ that i_k would be assigned if $(i_1, \dots, i_{n-2})$ determined the allocation. This means that $x\ P_{IY}(i_k)\ f_{i_k}(P_{IY})$. Case 1: for all $i \in I \backslash \{i_1, \dots, i_k\}$, $f_i(P_{IY}) \neq x$. Define $J = \{i_k\}$, $Z = \{y \in Y\colon$ there is no $j \in I$ such that $f_i(P_{IY}) = y\}$ and $P_{JZ} = P_{IY}|_{JZ}$. By successive application of CON, $f_{i_k}(P_{JZ}) = f_{i_k}(P_{IY}) \neq x$. Since $x \in Y$ and $x\ P_{JZ}(i_k)\ f_{i_k}(P_{JZ})$, PAR$_1$ is violated. Case 2: for some $i \in I \backslash \{i_1, \dots, i_k\}$, $f_i(P_{IY}) = x$. Define J

$= \{i_k, i\}$, $Z = \{f_{i_k}(P_{IY}), f_i(P_{IY})\}$ and $P_{JZ} = P_{IY}|_{JZ}$. By successive application of CON, $f_i(P_{JZ}) = f_i(P_{IY}) = x$. By (1), $w(i_k) > w(i)$, which implies that no coalition structure C_J on J makes $f(P_{JZ})$ stable with respect to w, C_J and P_{JZ}. This contradicts STA.■

Proposition 3.6. For $n > m \geq 2$, if allocation rule $f : D \rightarrow A$ satisfies STA, CON and PAR_1 then f has a hierarchy$_{n-2}$ of dictators.

Proof. Assume that f satisfies PAR_1, CON and STA, with power function w. Choose a subset $Y = \{x_1, x_2\}$ of X and define $Q_{NY} \in D$ to be such that, for all $i \in N$ and $k \in \{1, 2\}$, x_k is the kth most preferred object of X in $Q_{NY}(i)$. For $k \in \{1, 2\}$, let $i_k \in N$ satisfy $f_{i_k}(Q_{NY}) = x_k$. By Lemma 3.1, for all $i \in N \backslash \{i_1\}$, $w(i_1) > w(i)$.

If $n > 3$ then, taking this as the base of an induction argument, assume that there is $r \in \{2, \ldots, n-3\}$ such that: (i) $w(i_1) > w(i_2) > \ldots > w(i_r)$ and, for all $i \in N \backslash \{i_1, \ldots, i_{r-1}\}$, $w(i_{r-1}) > w(i)$; and (ii) letting $I = N \backslash \{i_1, \ldots, i_{r-2}\}$, with $I = N$ if $r = 2$, and $Q_{IY} = Q_{NY}|_{IY}$, $f_{i_{r-1}}(Q_{IY}) = x_1$ and $f_{i_r}(Q_{IY}) = x_2$. It must be shown that:

(a) for all $i \in N \backslash \{i_1, \ldots, i_r\}$, $w(i_r) > w(i)$; and
(b) letting $J = N \backslash \{i_1, \ldots, i_{r-1}\}$ and $Q_{JY} = Q_{NY}|_{JY}$, there is $i_{r+1} \in N \backslash \{i_1, \ldots, i_r\}$ such that $f_{i_r}(Q_{JY}) = x_1$ and $f_{i_{r+1}}(Q_{JY}) = x_2$.

To this end, choose $j \in N \backslash \{i_1, \ldots, i_r\}$. Define $H = \{i_r, j\}$, $Z = \{x_2\}$ and $Q_{HZ} = Q_{IY}|_{HZ}$. By CON, $f_{i_r}(Q_{IY}) = x_2$ implies $f_{i_r}(Q_{HZ}) = x_2$. By STA, $w(j) \leq w(i_r)$. Consequently,

$$\text{for all } i \in N \backslash \{i_1, \ldots, i_r\}, w(i) \leq w(i_r). \tag{2}$$

Consider now Q_{JY}. If $f_{i_r}(Q_{JY}) \neq x_1$ then, for some $i \in N \backslash \{i_1, \ldots, i_r\}$, $f_i(Q_{JY}) = x_1$. By Lemma 3.1, $w(i) > w(i_r)$, contradicting (2). Therefore, $f_{i_r}(Q_{JY}) = x_1$. This means that, for some $i_{r+1} \in N \backslash \{i_1, \ldots, i_r\}$, $f_{i_{r+1}}(Q_{JY}) = x_2$, so (b) is proved. To prove (a), choose $i \in N \backslash \{i_1, \ldots, i_r, i_{r+1}\}$. Define $K = \{i_r, i_{r+1}, i\}$ and $Q_{KY} = Q_{JY}|_{KY}$. By CON, $f_{i_r}(Q_{JY}) = x_1$ and $f_{i_{r+1}}(Q_{JY}) = x_2$ imply $f_{i_r}(Q_{KY}) = x_1$ and $f_{i_{r+1}}(Q_{KY}) = x_2$. By Lemma 3.1, $w(i_r) > w(i_{r+1})$ and $w(i_r) > w(i)$, which proves (a).

To sum up, there is a sequence $(i_1, \ldots, i_{n-2})$ of different members of N such that $w(i_1) > \ldots > w(i_{n-2})$ and, for all $i \in N \backslash \{i_1, \ldots, i_{n-2}\}$, $w(i_{n-2}) > w(i)$. To show that $(i_1, \ldots, i_{n-2})$ is a hierarchy$_{n-2}$ of dictators, proceed as in the proof of Proposition 3.5, by choosing $P_{IY} \in D$ and assuming that $f(P_{IY})$ does not agree with any allocation that would generate $(i_1, \ldots, i_{n-2})$ as a hierarchy$_{n-2}$ (in case 2, $f_{i_k}(P_{IY}) = \varnothing$ is possible and $w(i_k) > w(i)$ does not follow from (1) but from the preceding result that $w(i_1) > \dots > w(i_{n-2})$ and, for all $i \in N \backslash \{i_1, \dots, i_{n-2}\}$, $w(i_{n-2}) > w(i)$).■

Proposition 3.7. Allocation rule $f: D \to A$ satisfies STA*, CON and PAR_1 if, and only if, f has a hierarchy of dictators.

Proof. "⇐" If f has a hierarchy of dictators $(i_1, \dots, i_n)$ then it is easy to verify that f satisfies CON and PAR_1. And f satisfies STA* with power* function w such that, for all $k \in \{1, \dots, n\}$, $w(i_k) = 2^{n-k}$.

"⇒" Case 1: $n = 1$. Nothing to prove by PAR_1. Case 2: $n = 2$. Let $N = \{i, j\}$. By definition of power* function, either $w(i) > w(j)$ or $w(j) > w(i)$. Suppose $w(i) > w(j)$. It must be shown that, for all $P_{IY} \in D$ with $i \in I$, i is assigned his most preferred object x in Y according to $P_{IY}(i)$. This follows from PAR_1 if $I = \{i\}$, so let $I = N$. If $f_i(P_{IY}) \neq x$ then $f_j(P_{IY}) = x$. As $w(i) > w(j)$, no coalition structure C_N on N makes $f(P_{IY})$ stable with respect to w, C_N and P_{IY}, contradicting STA*.

Case 3: $n \geq 3$. Case 3a: $m = 1$. In this case, there is only one $P_{NX} \in D$. Hence, there is $i_1 \in N$ such that $f_{i_1}(P_{NX}) \neq \varnothing$. By CON, i_1 constitutes a hierarchy_1. Suppose that, for $k \in \{1, \dots, n-1\}$, a hierarchy_k $(i_1, \dots, i_k)$ has been defined. Let $I = N \backslash \{i_1, \dots, i_k\}$ and $P_{IX} = P_{NX}|_{IX}$. Then there is $i_{k+1} \in I$ such that $f_{i_{k+1}}(P_{IX}) \neq \varnothing$. By CON, $(i_1, \dots, i_k, i_{k+1})$ is a hierarchy_{k+1}. Case 3b: $m \geq 2$. Since STA* implies STA, by Propositions 3.5 and 3.6, f has a hierarchy_{n-2} of dictators $(i_1, \dots, i_{n-2})$. Let $N \backslash \{i_1, \dots, i_{n-2}\} = \{i, j\}$. As shown in the proofs of Propositions 3.5 and 3.6, for all $k \in \{i_1, \dots, i_{n-2}\}$, $w(k) > w(i)$ and $w(k) > w(j)$. Given this, by definition of power* function, either $w(i) > w(j)$ or $w(j) > w(i)$. Suppose $w(i) > w(j)$. It must be then shown that f has the hierarchy of dictators $(i_1, \dots, i_{n-2}, i, j)$. By CON, it is enough to prove that, for all $P_{IY} \in D$ such that $I = \{i, j\}$, i is assigned his most preferred object x in Y according to $P_{IY}(i)$. The proof is identical to that of case 2.■

References

Bogomolnaia, A., R. Deb and L. Ehlers (2005): "Strategy-proof assignment on the full preference domain", *Journal of Economic Theory* 123, 161–186.

Ehlers, L. (2002): "Coalitional strategy-proof house allocation", *Journal of Economic Theory* 105, 298–317.

Ehlers, L. and B. Klaus (2003): "Resource-monotonicity for house allocation problems", *International Journal of Game Theory* 32, 545–560.

Ehlers, L. and B. Klaus (2006): “Efficient priority rules”, *Games and Economic Behavior* 55, 372–384.

Ergin, H. İ. (2000): “Consistency in house allocation problems”, *Journal of Mathematical Economics* 34, 77–97.

Pápai, S. (2000a): “Strategyproof assignment by hierarchical exchange”, *Econometrica* 68, 1403−1433.

Pápai, S. (2000b): “Strategyproof multiple assignment using quotas”, *Review of Economic Design* 5, 91−105.

Pápai, S. (2001): “Strategyproof and nonbossy multiple assignments”, *Journal of Public Economic Theory* 3, 257–271.

Pápai, S. (2003): “Strategyproof exchange of indivisible goods”, *Journal of Mathematical Economics* 39, 931−959.

Shapley, L. and H. Scarf (1974): “On cores and indivisibility”, *Journal of Mathematical Economics* 1, 23–37.

Svensson, L.-G. (1994): “Queue allocation of indivisible goods”, *Social Choice and Welfare* 11, 323–330.

Svensson, L.-G. (1999): “Strategy-proof allocation of indivisible goods”, *Social Choice and Welfare* 16, 556–567.

Thomson, W. (1990): “The consistency principle”, in *Game Theory and Applications*, T. Ichiishi. A. Neyman and Y. Tauman, eds., Academic Press, San Diego, 187–215.