Allocation by coercion

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Abstract
The problem of allocating indivisible goods is considered when groups of individuals can make use of their power to plunder other groups. A monarch in a group of individuals is an individual who always obtains one of his most preferred goods. A Paretian condition together with a requirement of robust stability lead to the existence of monarchs in all subsets of individuals, except possibly one.

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1. Introduction

The typical economic model presumes that the activities carried out by the agents of the model are voluntary, in the sense that the will of the individuals is not forced. This approach neglecting the role of power in the allocation of resources contrasts with the pervasiveness of transactions governed by coercion, even in the presence of institutions entrusted to prevent, prosecute and punish the use of coercion.

A recent exception is Piccione and Rubinstein (2007), who study the existence and efficiency of the general equilibrium of an exchange economy with divisible goods in which power dictates who gets what. This paper also suggests a model of involuntary exchange. There are two differences with respect to the model analyzed by Piccione and Rubinstein. On the one hand, goods are indivisible. And, on the other, it is not only individuals that may exercise coercion, but also groups of individuals.

Specifically, the allocation model of indivisible goods by Shapley and Scarf (1974) is adopted, with the only difference that property rights are not defined. Since only power can guarantee that property rights will be respected, it seems more appropriate to consider such rights an output of the allocation process rather than an input. In addition, a number is supposed to measure the power of any coalition of individuals. An allocation α is stable with respect to that measure of power, the preferences of the individuals, and a certain coalition structure if no allocation β, no coalition K, and no group $G \cup J$ of individuals exist, with G consisting of members of K and J consisting of individuals not in K, such that: (i) the group $G \cup J$ has more power than the residual coalition $K \backslash G$; (ii) the set of goods that the members of $K \cup J$ receive in β coincides with the set of goods that they receive in α, so the members of $G \cup J$ can distribute among themselves their own goods together with those from the members of $K \backslash G$; and (iii) each member of $G \cup J$ prefers the good he receives in β to the one received in α.

A society is a set of individuals. A monarch in society I is an individual who, for every allocation problem involving I, always receives one of his most preferred goods. It is shown that every society has a monarch if the allocation must be Paretian and stable with respect to at least two coalition structures different from the trivial coalition structure, the one in which all the individuals belong to the same coalition. This result requires that the maximum number of individuals n in an allocation problem and the maximum number m of goods to be allocated in an allocation problem satisfy $m + 1 \geq n \geq 3$, so monarchs need abundance, and enough people, to arise.

2. Definitions and assumptions

The finite set $N = \{1, \dots, n\}$, consisting of the first $n \geq 1$ positive integers, is a set of names for individuals. A society is a non-empty subset of N. The finite set X contains $m \geq 1$ elements representing objects or anything that can be assigned to the individuals (like social status or public offices). The null object is represented by $0 \notin X$. For non-empty $Y \subseteq X$, T_Y is the set of preferences on Y, defined to be the set of reflexive, complete and transitive binary relations that can be defined on $Y \cup \{0\}$ such that every $x \in Y$ is strictly preferred to 0. For non-empty $Y \subseteq X$, L_Y is the set of strict preferences on $Y \subseteq X$, that is, the set of preferences on Y in which the underlying binary relation is antisymmetric. For society I and non-empty $Y \subseteq X$, a preference profile based on I and Y is a function $P_{IY} : I \to T_Y$. The set D is the set of all functions $P_{IY} : I \to T_Y$, where $I \subseteq N$ and $Y \subseteq X$.

Let $I \subseteq N$ and $Y \subseteq X$ be non-empty. If Y has as many members as I, then an allocation based on I and Y is an injective mapping $\alpha_{IY} : I \to Y$. If Y has less members than I, then an allocation based on I and Y is a mapping $\alpha_{IY} : I \to Y \cup \{0\}$ such that, for some $J \subset I$ having the same number of members as Y: (i) for all $i \in I \backslash J$, $\alpha_{IY}(i) = 0$; and (ii) for some allocation α_{JY} based on J and Y, $\alpha_{IY}(i) = \alpha_{JY}(i)$ for all $i \in J$. If $\alpha_{IY}(i) = 0$, i is assigned no object. If $\alpha_{IY}(i) \neq 0$, $\alpha_{IY}(i)$ is the member of Y that is assigned to individual $i \in I$. The set A is the set of all allocations based on I and Y, where $I \subseteq N$ and $Y \subseteq X$.

Definition 2.1. An allocation rule on non-empty $E \subseteq D$ is a mapping $f : E \to A$.

An allocation rule is a mapping f that transforms each profile P_{IY} from a subset E of the set D of all preference profiles into an allocation $f(P_{IY})$ based on I and Y. If $i \in I$ is assigned some object in $f(P_{IY})$, then $f_i(P_{IY})$ designates that object.

Definition 2.2. For allocation rule $f : E \to A$, individual $i \in I$ is a monarch in society I if, for all $P_{IY} \in E$, no member of Y is strictly preferred to $f_i(P_{IY})$ in $P_{IY}(i)$.

Definition 2.3. Allocation rule $f : E \to A$ is: (i) monarchical if there is a list $(i_1, \dots, i_n)$ of the n members of N such that, for each society I, the member of I appearing first in the list $(i_1, \dots, i_n)$ is a monarch in I; and (ii) essentially monarchical if it is monarchical except, possibly, for society $\{i_{n-1}, i_n\}$, that may not have a monarch.

Definition 2.4. Allocation rule $f : E \to A$ is weakly Paretian if, for all $P_{IY} \in E$ and $i \in I$, there is no $x \in \{y \in Y$: for all $j \in I$, $f_j(P_{IY}) \neq y\}$ such that i prefers x to $f_i(P_{IY})$ in $P_{IY}(i)$.

Definition 2.5. A power function is a mapping $w : 2^N\backslash\{\varnothing\} \rightarrow \mathbb{N}\backslash\{0\}$ satisfying, for each $S \in 2^N\backslash\{\varnothing\}$ and each $i \in N\backslash S$, $w(S \cup \{i\}) > w(S)$.

A power function assigns, to each non-empty group S of individuals, a positive integer $w(S)$ that measures the power of S. The definition of power function presumes that the addition of an individual to any group of individuals increases the power of the group. If, for some $i \in N$, $S = \{i\}$, then $w(i)$ abbreviates $w(\{i\})$.

Definition 2.6. For society I, a coalition structure C on I is a set $\{I_1, \dots, I_k\}$ such that: (i) for all $r \in \{1, \dots, k\}$, $I_r \neq \varnothing$; (ii) for all $r \in \{1, \dots, k\}$ and $t \in \{1, \dots, k\}\backslash\{r\}$, $I_r \cap I_t = \varnothing$; and (iii) $I = I_1 \cup \dots \cup I_k$.

A coalition structure on a society I is just a partition of I. A coalition structure is non-trivial if it contains at least two members.

Definition 2.7. An allocation α_{IY} based on I and Y is stable with respect to power function w, coalition structure C on I and preference profile P_{IY} based on I and Y if there do not exist $K \in C$, non-empty $G \subset K$, $J \subseteq I\backslash K$ (where J can be the empty set), and allocation β_{IY} based on I and Y such that:

- (i) $w(G \cup J) > w(K\backslash G)$;
- (ii) for all $i \in G \cup J$, $\beta_{IY}(i)$ is strictly preferred to $\alpha_{IY}(i)$ in $P_{IY}(i)$; and
- (iii) for all $i \in I\backslash(K \cup G)$, $\beta_{IY}(i) = \alpha_{IY}(i)$.

An allocation α_{IY} is stable with respect to power function w, coalition structure C and preference profile P_{IY} if no coalition K in the structure C can be plundered. This means that no subset G of K can join forces with members J outside K to create a coalition $G \cup J$ having more power, according to w, than the residual coalition $K\backslash G$ and use that power to reallocate the objects that α_{IY} assigns to the members of $K \cup J$ in such a way that all the members of $G \cup J$ are better off, according to the preferences in P_{IY}.

Definition 2.8. An allocation α_{IY} based on I and Y, with I having at least three members, is minimally robust with respect to power function w, family F_I of coalition structures on I and preference profile P_{IY} based on I and Y if there are at least two non-trivial coalition structures C' and C'' in F_I such that, for all $C \in \{C', C''\}$, α_{IY} is stable with respect to w, C and P_{IY}.

Definition 2.9. Allocation rule $f : E \to A$ is minimally robust if there is a power function w satisfying the following: for each $P_{IY} \in E$, there is a family F_I of coalition structures on I such that (i) if I has at least three members, then $f(P_{IY})$ is minimally robust with respect to w, F_I, and P_{IY}, and (ii) if I has two members, then, for some $C \in F_I$, $f(P_{IY})$ is stable with respect to w, C, and P_{IY}.

When power mediates allocation, it is to be expected that coalitions of two types will be formed. There will be, on the one hand, defensive coalitions, established to prevent other coalitions from snatching what the members of the defensive coalition possess. But, on the other hand, there will be aggressive coalitions, constituted precisely to snatch what other coalitions have. The interplay between these two kinds of coalitions would probably reach a steady state, in which defensive coalitions are too powerful to be plundered or in which aggressive coalitions have achieved their goals. Both situations are characterized by the fact that there is no need or possibility to exercise power. An allocation rule is intended to capture such resting points. A minimally robust allocation rule implicitly looks for power functions that can lead to an equilibrium in the use of power, asking for the existence of at least two coalition non-trivial structures under which there is no need or opportunity to resort to power to alter the allocation.

3. Result

Proposition 3.1. With $m \geq n - 1$ and $n \geq 3$, let $f : E \to A$ be an allocation rule such that $E \subseteq D$ and (1) holds. If f is weakly Paretian and minimally robust, then f is essentially monarchical.

For each society I with at least two members, there are $Y \subseteq X$ having one member less than I, $P_{IY} \in E$, and $p \in L_Y$ such that, for all $i \in I$, $P_{IY}(i) = p$. 1)

Proof. Let w be the power function under which f is minimally robust.

Step 1: for every society I there is $i \in I$ such that $w(I\backslash\{i\}) \leq w(i)$. Let I have r members. The claim is obviously true for $r = 1$, so let $r \geq 2$. By (1), there is $Y = \{x_1, \ldots, x_{r-1}\} \subseteq X$ and $P_{IY} \in E$ such that, for all $i \in I$ and all $k \in \{1, \ldots, r - 1\}$, x_k is the kth most preferred object in $P_{IY}(i)$. Being f weakly Paretian, all objects in Y are assigned to some member of I. Hence, for some $i \in I$, $f_i(P_{IY}) = x_1$. For $k \in \{2, \ldots, r - 1\}$, let $i_k \in I$ be assigned x_k in $f(P_{IY})$. By minimal robustness, $f(P_{IY})$ must be stable with respect to w, P_{IY} and some coalition structure C on I. Let β_{IY} be the allocation such that, for all $k \in \{2, \dots, r\}$, $\beta(i_k) = x_{k-1}$. By the assumption that receiving an object is strictly preferred to receiving none, for all $j \in I\backslash\{i\}$, $\beta_{IY}(j)\ P_{IY}(j)\ f_j(P_{IY})$. In view of this, by stability of $f(P_{IY})$, $w(I\backslash\{i\}) \leq w(i)$.

Step 2: there is a list $(i_1, \dots, i_n)$ of the n members of N such that $w(i_1) > w(i_2) > \dots > w(i_{n-2}) > w(i_{n-1}) \geq w(i_n)$. Consider society N. By step 1, for some $i_1 \in N$, $w(i_1) \geq w(N\backslash\{i_1\})$. By definition of power function, $w(i_1) \geq w(N\backslash\{i_1\})$ implies that, for all $j \in N\backslash\{i_1\}$, $w(i_1) > w(j)$. Arguing inductively, suppose that, for some $k < n - 2$, $(i_1, \dots, i_k)$ are such that $w(i_1) > \dots > w(i_k)$ and, for all $j \in N\backslash\{i_1, \dots, i_k\}$, $w(i_k) > w(j)$. It must be shown that, for some $i_{k+1} \in N\backslash\{i_1, \dots, i_k\}$, $w(i_1) > \dots > w(i_{k+1})$ and, for all $j \in N\backslash\{i_1, \dots, i_k, i_{k+1}\}$, $w(i_{k+1}) > w(j)$. By step 1, for some $i_{k+1} \in N\backslash\{i_1, \dots, i_k\}$, $w(i_{k+1}) \geq w(N\backslash\{i_1, \dots, i_{k+1}\})$. Since $k < n - 2$, the set $N\backslash\{i_1, \dots, i_{k+1}\}$ has at least two members. Hence, by definition of power function, $w(i_{k+1}) \geq w(N\backslash\{i_1, \dots, i_{k+1}\})$ implies that, for all $j \in N\backslash\{i_1, \dots, i_{k+1}\}$, $w(i_{k+1}) > w(j)$. This proves that members $i_1, \dots, i_{n-2}$ of N can be found such that $w(i_1) > \dots > w(i_{n-2})$ and both $w(i_{n-2}) > w(i_{n-1})$ and $w(i_{n-2}) > w(i_n)$. As regards, i_{n-1} and i_n, it may be that $w(i_{n-1}) = w(i_n)$.

Step 3: f is essentially monarchical. By step 2, there is a list $(i_1, \dots, i_n)$ of the n members of N such that $w(i_1) > w(i_2) > \dots > w(i_{n-2}) > w(i_{n-1}) \geq w(i_n)$. Choose society $I \neq \{i_{n-1}, i_n\}$. Let i be the member of I appearing first in the list $(i_1, \dots, i_n)$. It must be shown that i is a monarch in I. If $I = \{i\}$, then, since f is weakly Paretian, i is a monarch in I. If I has at least two members, then $I \neq \{i_{n-1}, i_n\}$ implies that $i \notin \{i_{n-1}, i_n\}$. Case 1: I has two members. Let j be the other member of I. Given that i appears before j in $(i_1, \dots, i_n)$ and $i \notin \{i_{n-1}, i_n\}$, $w(i) > w(j)$. By minimal robustness, i must be a monarch in $\{i, j\}$. Case 2: I has more than two members. Suppose i is not a monarch in I. By step 1, there is $j \in I$ such that $w(j) \geq w(I\backslash\{j\})$. By definition of power function, $w(j) \geq w(I\backslash\{j\})$ implies $w(j) > w(i)$. This means that $j \in I$ must appear before i in the list $(i_1, \dots, i_n)$: contradiction.■

Proposition 3.1 does not hold for $m < n - 1$. The reason is that step 1 need not be true, because it is possible to have, for all $i \in I$, $w(I\backslash\{i\}) > w(i)$ due to the lack of enough objects to improve the position of every individual in $I\backslash\{i\}$ when the members of $I\backslash\{i\}$ try to allocate the objects without taking into account i. This fact has an interesting interpretation: the concentration of allocation power in the hands of a monarch requires society to be endowed with a sufficiently high amount of wealth. In other words, scarcity is an antidote for monarchs.

Proposition 3.1 can be easily transformed into a result establishing sufficient conditions for a monarchical allocation rule: just modify the definition of power function so that there is some $i \in N$ such that, for all $j \in N \setminus \{i\}$, $w(i) < w(j)$. This says that there exists a least powerful individual. This condition ensures that society $\{i_{n-1}, i_n\}$ will also have a monarch.

Proposition 3.1 does not restrict the way the objects left by the monarch i of society I (with at least three members) are allocated among the rest of individuals in $I \setminus \{i\}$. In fact, the reasoning in the proof of Proposition 3.1 cannot be applied to the subsociety $I \setminus \{i\}$, because i, being at least as powerful as the rest of the society, can essentially determine how the rest of objects are assigned. As an illustration, let I have $r \geq 4$ members and i be the monarch in I. Choose any group J of $I \setminus \{i\}$ having $r - 3$ members. Then, with $\{j, k\} \subseteq I \setminus \{i\}$, the non-trivial coalition structures $\{J \cup \{i\}, \{j, k\}\}$ and $\{J \cup \{i\}, \{j\}, \{k\}\}$ can make any allocation among the members of J minimally robust. Consequently, the inclusion of the monarch in some coalition prevents the emergence of any hierarchical structure among the rest of individuals. That structure could be generated by imposing a non-bossiness condition.

Non-bossiness, as defined in Svensson (1999) or Pápai (2000), is a condition expressed in terms of preferences: if, by changing his preferences, an individual induces a change in the allocation, then he must be affected by this allocation change. In the present model, non-bossiness would be a condition restricting the set of admissible coalition structures with respect to which stability is defined: admissible coalition structures should be such that individuals that are "too powerful" must constitute singleton coalitions. Such non-bossy coalition structures are motivated by a principle of non-interference: individuals powerful enough to get what they want by themselves should refrain from meddling in the allocation of objects among non-powerful individuals. With minimal robustness defined in terms of non-bossy coalition structures, every society with $r \geq 3$ individuals would allocate objects following a fixed hierarchy of $r - 2$ individuals; see Ergin (2000) for another set of sufficient conditions for a partial hierarchy to emerge. By including the condition that some individual is the least powerful individual in the whole society N, a complete hierarchy would be present in every society.

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