Three simple axiomatizations of the Hirsch index

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Abstract

The Hirsch index is a number that synthesizes a researcher's output. It is defined as the maximum number *h* such that the researcher has *h* papers with at least *h* citations each. Woeginger (2008) suggests an axiomatic characterization of the Hirsch index that: (i) requires five axioms; and (ii) holds for indices taking values in the set of non-negative integers. This note suggests three characterizations, each of them requiring just two axioms and one of them valid for indices taking values in the set of non-negative real numbers.

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1. Introduction

The Hirsch (2005) index of a researcher is the maximum number h of papers of the researcher having at least h citations each; see Wikipedia (2008) for a discussion of advantages and criticisms of the Hirsch index. This index can be interpreted as a measure of the productivity of a researcher or as a measure of the impact or quality of a researcher's output. Woeginger (2008) suggests five axioms to characterize the Hirsch index for the case in which indices are assumed to yield non-negative integer values. This note presents three additional characterizations of the Hirsch index. All of them hinge of a type of axiom indicating what is necessary and sufficient for another paper to increase the index; see A2 and B2 in Section 2. In particular, B2 characterizes the Hirsch index in the domain of surjective indices taking values in the set of non-negative integers. Each of the three characterizations invokes just two axioms. The last characterization improves upon Woeginger's in being valid for the case in which indices can take values in the set of non-negative real numbers.

2. Definitions and axioms

Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{R}$ the set of non-negative real numbers. Members of $\mathbb{N}$ represent both the number of papers of a given researcher and the number of citations that a paper can receive. Define X to be the set of all vectors $x = (x_1, x_2, \dots, x_n)$ such that $n \in \mathbb{N}\backslash\{0\}$ and $x_1 \geq x_2 \geq \dots \geq x_n$. For $x \in X$: (i) d_x is the number of components of vector x (the dimension of x); (ii) c_x is the number of components of vector x different from 0; and (iii) for $i \in \{1, \dots, d_x\}$, x_i is the ith component of vector x and stands for the total number of citations of paper i. With $\varnothing$ designating the empty vector, a researcher's output will be represented by a member of $D = X \cup \{\varnothing\}$. For $x = \varnothing$ the convention is that $c_x = d_x = 0$ and that, for all, $n \in \mathbb{N}$, $n \geq \varnothing$. For $n \in \mathbb{N}$, define $D_n = \{x \in D: d_x = n\}$ to be the set of outputs consisting of n papers.

Definition 2.1. An index$_1$ is a mapping $f: D \to \mathbb{N}$. An index$_2$ is a mapping $f: D \to \mathbb{R}$.

Definition 2.2. The Hirsch index is both the index$_1$ and index$_2$ h such that $h(\varnothing) = 0$ and, for all $x \in X$, $h(x) = max\{n \in \{0, 1, \dots, c_x\}: x_n \geq n\}$.

A1. For all $x \in D$, $f(x) \leq d_x$.

A1 sets an upper bound to the index: the number of papers in an output determines the maximum value that can be ascribed to the output. A1 suggests that the units of measure of the index are papers. For $n \in \mathbb{N}$, $x \in D_n$ and $a \in \mathbb{N}$, $(x \oplus a)$ is the vector $y \in D_{n+1}$ such that $y_{n+1} = a$ and, for all $i \in \{1, \dots, n\}$, $y_i = x_i$. By definition, $\varnothing \oplus \varnothing = \varnothing$ and, for all $x \in X$, $\varnothing \oplus x = x \oplus \varnothing = x$.

A2. For all $x \in D$, $a \in \mathbb{N} \cup \{\varnothing\}$ and $b \in \mathbb{N} \cup \{\varnothing\}$ such that $(x \oplus a) \in D$ and $(x \oplus b) \in D$, $f(x \oplus a) > f(x \oplus b)$ if, and only if, $a > d_x \geq b$.

A2 establishes what is necessary and sufficient for the citations of another paper to increase the index. Suppose that a paper is added to output x and that this paper does not receive more citations that the least cited paper. Consider two situations: in one, the paper receives a citations and, in the other, receives b. Then output $(x \oplus a)$ has a higher index that output $(x \oplus b)$ if, and only if: (i) a is greater than the number d_x of papers in x (so the new paper must receive at least one citation for each paper in the initial output x); and (ii) b cannot be greater than the number d_x of papers in x.

It may appear that A2 is just what the Hirsch index says, so the step from A2 to the Hirsch index would be rather short. Even if this is the case, it might still be interesting to consider A2 in order to ascertain how short is that step. It will nonetheless be argued that the distance between A2 and the Hirsch index is not so short as it may at first appear, because A2 fails to imply four of the five axioms in Woeginger's (2008, p. 228) characterization. These are listed below as W1, W2, W3, W4 and W5.

W1. For all $x \in D$, such that $d_x = 0$ or $c_x = 0$, $f(x) = 0$.

W2. For all $x \in D$ and $y \in D$, $x \geq y$ implies $f(x) \geq f(y)$.

W3. For all $x \in D$, $f(x \oplus f(x)) \leq f(x)$.

W4. For all $x \in X$, $i \in \{1, \dots, d_x\}$ and $y_i \in \mathbb{N}$, if $y_i > x_i$ and $(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_{d_x}) \in D$ then $f(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_{d_x}) \leq f(x) + 1$.

W5. For all $x \in D$ and $k \in \mathbb{N}\backslash\{0\}$, if $(x_1 + k, x_2 + k, \dots, x_{d_x} + k, f(x) + k) \in D$ then $f(x_1 + k, x_2 + k, \dots, x_{d_x} + k, f(x) + k) > f(x)$.

Remark 2.3. For $\alpha \in \{\text{W1, W2, W3, W4}\}$, A2 does not imply α.

The index f such that, for all $x \in D$, $f(x) = h(x) + 1$ satisfies A2 but does not satisfy W1 (since $f(\varnothing) = 1$) nor W3 (with $x = (3, 3)$, $f(x) = 3$ and $4 = f(3, 3, 3) = f(x \oplus f(x)) > f(x) = f(3, 3) = 3$). The index f such that, for all $x \in D$, $f(x) = 2h(x)$ satisfies A2 but does not satisfy W4 (because $4 = f(2, 2) > 1 + f(2, 1) = 3$). With $D' = \{x \in X\colon x_1 = 4 \text{ and } x_2 = x_3 = 3\}$, the index f such that, for all $x \in D'$, $f(x) = 4$ and, for all $x \in D \backslash D'$, $f(x) = h(x)$ satisfies A2 but does not satisfy W2 (as $3 = f(4, 4, 3) < f(4, 3, 3) = 4$).

Remark 2.4. A2 implies W5.

Let $x \in D_n$. If $x_1 \geq 1$, define $i \in \{1, \dots, n\}$ to be the largest r such that $x_r > r - 1$; otherwise, $i = 0$. Case 1: $i = 1$. Then, by A2, $f(\varnothing) < f(x_1) < f(x_1, x_2) < \dots < f(x_1, \dots, x_i) = f(x_1, \dots, x_i, x_{i+1}) = \dots = f(x)$. Hence, $f(x) \geq i$. Case 1a: $i < n$. If $(x_1 + k, x_2 + k, \dots, x_n + k, f(x) + k) \in D$ then $x_{i+1} \geq f(x) \geq i$. By definition of i, $x_{i+1} \leq i$. In sum, $x_{i+1} = i$ and $f(x) = i$. Given this, the largest $r \in \{1, \dots, n\}$ such that $x_r + k > r - 1$ in $(x_1 + k, x_2 + k, \dots, x_n + k, f(x) + k)$ is at least $i + 1$. This, by A2, implies that $f(x_1 + k, x_2 + k, \dots, x_n + k, f(x) + k) \geq i + 1 > f(x)$. Case 1b: $i = n$. In this case, the largest $r \in \{1, \dots, n\}$ such that $x_r + k > r - 1$ in $(x_1 + k, x_2 + k, \dots, x_n + k, f(x) + k)$ is $n + 1$ and the reasoning is as in case 1a. Case 2: $i = 0$. This means that $x = \varnothing$ or $x = (0, \dots, 0)$. Case 2a: $x = (0, \dots, 0)$. If $(k, \dots, k, f(x) + k) \in D$ then $f(x) = 0$. By repeated application of A2, $f(k, \dots, k) \geq f(k) > f(0)$, so $f(k, \dots, k) > 0 = f(x)$. Case 2b: $x = \varnothing$. By A2, $f(f(\varnothing) + k) > f(\varnothing)$.

B1. For every $n \in \mathbb{N}$ there is $x \in D$ such that $f(x) = n$.

As a property of an index$_1$, B1 asserts that the index$_1$ is surjective: for every possible value of the index$_1$, some output is assigned that value.

B2. For all $x \in D$, $y \in D$, $a \in \mathbb{N} \cup \{\varnothing\}$ and $b \in \mathbb{N} \cup \{\varnothing\}$ such that $(x \oplus a) \in D$, $(y \oplus b) \in D$ and $f(x) = f(y)$, $f(x \oplus a) > f(y \oplus b)$ if, and only if, $a > d_x \geq b$.

A2 is the particular case of B2 in which $x = y$. In fact, B2 is the generalization of A2 motivated by the following idea: if the index f attributes the same value to two outputs x and y then, from the perspective of the index, both outputs can be considered equivalent and this motivates replacing with y one of the two occurrences of x in the term "$f(x \oplus a) > f(x \oplus b)$" in A2. In other words, if f does not distinguish x from y then f should not distinguish $(x \oplus b)$ from $(y \oplus b)$. The replacement of $(x \oplus b)$ by $(y \oplus b)$ in A2 when $f(x) = f(y)$ leads to B2.

B3. For every $n \in \mathbb{N}$ there is $x \in D_n$ such that $f(x) = n$.

B3 is similar to B2 but slightly more demanding: for every integer value n of an output, some output with size n is assigned value n. If $f(x)$ determines the number of "valuable" papers in output x then B3 states that, for every size n, some output with size n has n valuable papers.

3. Results

Remark 3.1. The Hirsch index satisfies A1, A2, B1, B2 and B3.

A1 and B1 are immediate implications of the definition of the Hirsch index. With respect to B3, if $n = 0$ then $\varnothing \in D_0$ and $h(\varnothing) = 0$; and if $n \in \mathbb{N}\backslash\{0\}$ then $x \in D_n$ such that, for all $i \in \{1, \dots , n\}$, $x_i = n$ satisfies $h(x) = n$. Since B2 implies A2, it suffices to show that the Hirsch index satisfies B2. Suppose first that $a > d_x \geq b$, $(x \oplus a) \in D$, $(y \oplus b) \in D$ and $h(x) = h(y)$. If $x = \varnothing$ then $a > 0$ and $b = 0$, in which case $h(x \oplus a) = 1$. In addition, $h(x) = h(y)$ implies $h(y) = 0$, so $h(y \oplus b) = 0$. If $x \neq \varnothing$ then, since $a > d_x$ and $(x \oplus a) \in D$, for all $i \in \{1, \dots , d_x\}$, $x_i > d_x$. This implies $h(x \oplus a) > h(x) = d_x$. On the other hand, $h(y) = h(x)$ implies $h(y) = d_x$. Thus, $d_x \geq b$ implies $h(y \oplus b) = h(y) = h(x)$.

Suppose next that $h(x \oplus a) > h(y \oplus b)$, $(x \oplus a) \in D$, $(y \oplus b) \in D$ and $h(x) = h(y)$. As the Hirsch index satisfies W2, $h(y \oplus b) \geq h(y)$. In view of this, $h(x \oplus a) > h(y) = h(x)$, which demands $a > h(x)$, so all papers in x have more than $h(x)$ citations. In consequence, $h(x) = d_x$: if $h(x) < d_x$ then paper d_x in x has less than d_x citations, for which reason paper $d_x + 1$ in $(x \oplus a)$ has also less than d_x citations and $h(x \oplus a)$ cannot be greater than $h(x)$. In conclusion, $h(x) = d_x$ yields $a > d_x$. It then remains to be shown that $b \leq d_x$. Since $h(x \oplus a) > h(x)$, $h(x \oplus a) = d_x + 1$. Given this, $h(y \oplus b) \leq d_x$. Moreover, $h(y \oplus b) \geq h(y) = h(x) = d_x$. Summing up, $h(y \oplus b) = d_x$. As $h(y) = d_x$, it must be that $b \leq d_x$.

Proposition 3.2. An index$_1$ f satisfies A1 and A2 if, and only if, f is the Hirsch index.

Proof. "$\Leftarrow$" Remark 3.1. "$\Rightarrow$" Let f be an index$_1$ satisfying A1 and A2. The proof is by induction on the sets D_n. Step 1: f agrees with the Hirsch index on $D_0 = \{\varnothing\}$. By definition of index, $f(\varnothing) \geq 0$. By A1, $f(\varnothing) \leq d_\varnothing = 0$. Therefore, $f(\varnothing) = 0 = h(\varnothing)$.

Step 2: f agrees with the Hirsch index on D_1. Let $x \in D_1$. By A1, $f(x) \leq d_x = 1$. By definition of index, $f(x) \geq 0$. Consequently, $f(x) \in \{0, 1\}$. Case 1: $x_1 = 0$. By A2, $f(\varnothing \oplus 0) > f(\varnothing \oplus \varnothing)$ if, and only if, $0 > d_\varnothing \geq \varnothing$. Since $d_\varnothing = c_\varnothing = 0$, $f(x) \leq f(\varnothing)$. By A2, $f(\varnothing \oplus$

$\varnothing) > f(\varnothing \oplus 0)$ if, and only if, $\varnothing > d_{\varnothing} \geq 0$. As $\varnothing \leq 0 = d_{\varnothing}$, $f(\varnothing) \leq f(x)$. To sum up, $f(x) = f(\varnothing) = 0 = h(0)$. Case 2: $x_1 \geq 1$. By A2, $f(\varnothing \oplus x_1) > f(\varnothing \oplus 0)$ if, and only if, $x_1 > d_{\varnothing} \geq 0$. Since $d_{\varnothing} = c_{\varnothing} = 0$, $f(x) > f(0)$. By case 1, $f(0) = 0$, so $f(x) > 0$. It then follows from $f(x) \in \{0, 1\}$ that $f(x) = 1 = h(x)$.

Step 3: for $n \in \mathbb{N}\backslash\{0, 1\}$, f agrees with the Hirsch index on D_n. Choose $n \in \mathbb{N}\backslash\{0, 1\}$ and, by steps 1 and 2, suppose that, for all $k \in \{0, 1, \ldots, n-1\}$, f agrees with the Hirsch index on D_k. To prove that f agrees with the Hirsch index on D_n, choose $x \in D_n$. With $y = (x_1, \ldots, x_{n-1})$, by the induction hypothesis, $f(y) = h(y)$. Additional papers or citations cannot lower the Hirsch index, so $h(x) \geq h(y)$. Case 1: $h(x) > h(y)$. Let $h(y) = k$. As $y \in D_{n-1}$, $k \leq n-1$. If $k < n-1$ then $x_{n-1} \leq k$. Being x a member of D, $x_{n-1} \leq k$ implies $x_n \leq k$. In view of this, $h(x) = k$ and the assumption $h(x) > k$ is contradicted. As a consequence, $k = n-1$. Given this, $h(x) > h(y)$ implies $h(x) \geq n$. Since the Hirsch index satisfies A1, $h(x) \leq d_x = n$. In sum, $h(x) = n$. By A2, $f(y \oplus x_n) > f(y \oplus \varnothing)$ if, and only if, $x_n > n-1 \geq 0$, where $d_y = c_y = n-1$. As $h(x) = n$, $x_n > n-1$. By the assumption that $n \in \mathbb{N}\backslash\{0, 1\}$, $n-1 \geq 0$. As a result, $f(y \oplus x_n) > f(y \oplus \varnothing)$; that is, $f(x) > f(y)$. Summarizing, $f(x) > f(y) = h(y) = n-1$. By A1, $f(x) \leq d_x = n$ and the final conclusion is $f(x) = n = h(x)$.

Case 2: $h(x) = h(y)$. By the induction hypothesis, $f(y) = h(y)$, so it suffices to show that $f(x) = f(y)$. It follows from $h(y) \leq n-1$ and $h(x) = h(y)$ that $x_n \leq n-1$. By A2, $f(y \oplus x_n) > f(y \oplus \varnothing)$ requires $x_n > n-1 \geq \varnothing$, which is not the case (if $c_y < d_y$ then $x_{n-1} \leq h(y) \leq c_y$ and, given this, $x_n \leq h(y)$, which shows that $x_n > c_y \geq \varnothing$ is neither the case). Accordingly, $f(y \oplus x_n) \leq f(y \oplus \varnothing)$; that is, $f(x) \leq f(y)$. In addition, by A2, $f(y \oplus \varnothing) > f(y \oplus x_n)$ demands $\varnothing > n-1 \geq x_n$, which is not the case (even if c_x replaced d_x in A2, because $c_x \geq 0$). Hence, $f(y \oplus \varnothing) \leq f(y \oplus x_n)$; that is, $f(y) \leq f(x)$.■

Remark 3.3. As hinted in the proof of Proposition 3.2, Proposition 3.2 holds if, in A2, d_x is replaced by c_x.

Remark 3.4. A1 is not redundant in Proposition 3.2, because the index_1 such that, for all $x \in D$, $f(x) = 1 + h(x)$ satisfies A2, does not satisfy A1 and is not the Hirsch index.

Remark 3.5. Example 3.6 shows that Proposition 3.2 does not hold if "index_1" is replaced by "index_2".

Example 3.6. With $k \in \mathbb{R}\backslash\{0\}$, let f be the index_2 such that, for all $x \in X$, $h(x) = 0$ implies $f(x) = 0$ and $h(x) \geq 1$ implies $f(x) = h(x) - ½$. Though f satisfies both A1 and A2, it is not the Hirsch index.

For $n \in \mathbb{N}$, define $H_n = \{x \in D\text{: } h(x) = n\}$. Note that $H_0 = \{x \in D\text{: } c_x = 0\}$ and, for all $n \in \mathbb{N}\backslash\{0\}$ and $k \in \{0, 1, \dots, n-1\}$, $H_n \cap D_k = \varnothing$.

Lemma 3.7. If index$_2$ f satisfies B2 then, for all $n \in \mathbb{N}$, $x \in H_n$ and $y \in H_n$, $f(x) = f(y)$.

Proof. Step 1: for all $x \in H_0$ and $y \in H_0$, $f(x) = f(y)$. By B2, $f(\varnothing \oplus 0) > f(\varnothing \oplus \varnothing)$ if, and only if, $0 > d_{\varnothing} \geq \varnothing$. Since $d_{\varnothing} = 0$, $f(0) \leq f(\varnothing)$. By B2, $f(\varnothing \oplus \varnothing) > f(\varnothing \oplus 0)$ if, and only if, $\varnothing > d_{\varnothing} \geq 0$. Since $\varnothing \leq d_{\varnothing}$, $f(\varnothing) \leq f(0)$. Therefore, $f(\varnothing) = f(0)$. For $n \in \mathbb{N}\backslash\{0\}$, let 0^n designate the vector consisting of n components all of which are zero. Taking $f(\varnothing) = f(0)$ as the base of an induction argument, choose $n \in \mathbb{N}\backslash\{0, 1\}$ and assume that, for all $k \in \{1, \dots, n-1\}$, $f(\varnothing) = f(0^k)$. The aim is to show that $f(\varnothing) = f(0^n)$. By B2, $f(0^{n-1} \oplus \varnothing) > f(0^{n-1} \oplus 0)$ if, and only if, $\varnothing > n - 1 \geq 0$. As $\varnothing \leq n - 1$, $f(0^{n-1}) \leq f(0^n)$. By B2, $f(0^{n-1} \oplus 0) > f(0^{n-1} \oplus \varnothing)$ if, and only if, $0 > n - 1 \geq \varnothing$. Given that $0 \leq n - 1$, $f(0^n) \leq f(0^{n-1})$. Hence, $f(0^n) = f(0^{n-1}) = f(\varnothing)$. Consequently, for all $x \in H_0$, $f(x) = f(\varnothing)$.

Step 2: for all $n \in \mathbb{N}\backslash\{0\}$, $x \in H_n$ and $y \in H_n$, $f(x) = f(y)$. Choose $n \in \mathbb{N}\backslash\{0\}$ and, by step 1, suppose that, for all $k \in \{0, 1, \dots, n-1\}$, $x \in H_k$ and $y \in H_k$, $f(x) = f(y)$. To prove that for all $x \in H_n$ and $y \in H_n$, $f(x) = f(y)$, choose $x \in H_n \cap D_n$ and let $y \in D_n$ satisfy, for all $i \in \{1, \dots, n\}$, $y_i = n$. For $r \in \mathbb{N}\backslash\{0\}$ and $z \in D_r$, $z_{-r} = (z_1, \dots, z_{r-1})$ is the vector obtained from z by deleting the last component. Since $h(x_{-n}) = h(y_{-n}) = n - 1$, by the induction hypothesis, $f(x_{-n}) = f(y_{-n}) = n - 1$. By B2, $f(x_{-n} \oplus x_n) > f(y_{-n} \oplus n)$ if, and only if, $x_n > n - 1 \geq n$. Therefore, $f(x) = f(x_{-n} \oplus x_n) \leq f(y_{-n} \oplus n) = f(y)$. By B2, $f(y_{-n} \oplus n) > f(x_{-n} \oplus x_n)$ if, and only if, $n > n - 1 \geq x_n$. As $x \in H_n$, $x_n > n - 1$. Accordingly, $f(y) \leq f(x)$. Summarizing, for all $z \in H_n \cap D_n$, $f(z) = f(y)$. Given this, choose $k \in \mathbb{N}\backslash\{0\}$ and, arguing inductively, assume that, for all $z \in H_n \cap D_n \cap \dots \cap D_{n+k-1}$, $f(z) = f(y)$. It must be shown that, for all $z \in H_n \cap D_{n+k}$, $f(z) = f(y)$. To this end, choose $x \in H_n \cap D_{n+k}$. By the induction hypothesis, $f(x_1, \dots, x_{n+k-1}) = f(y)$. By B2, $f((x_1, \dots, x_{n+k-1}) \oplus x_{n+k}) > f(y \oplus \varnothing)$ if, and only if, $x_{n+k} > n + k - 1 \geq \varnothing$. The fact that $x \in H_n$ implies $x_{n+k} \leq n$. In view of this, $x_{n+k} \leq n + k - 1$ and, hence, $f(x) = f((x_1, \dots, x_{n+k-1}) \oplus x_{n+k}) \leq f(y \oplus \varnothing) = f(y)$. By B2, $f(y \oplus \varnothing) > f((x_1, \dots, x_{n+k-1}) \oplus x_{n+k})$ if, and only if, $\varnothing > n + k - 1 \geq x_{n+k}$, so $f(y) \leq f(x)$. As a result, $f(x) = f(y)$.■

Proposition 3.8. An index$_1$ f satisfies B1 and B2 if, and only if, f is the Hirsch index.

Proof. "$\Leftarrow$" Remark 3.1. "$\Rightarrow$" Let f be an index$_2$ satisfying B1 and B2. Define $\alpha^0 = \varnothing$ and, for $n \in \mathbb{N}\backslash\{0\}$, define $\alpha^n \in D_n$ to be such that, for all $i \in \{1, \dots, n\}$, $x_i = n$.

Step 1: for all $n \in \mathbb{N}$, $f(\alpha^{n+1}) > f(\alpha^n)$. Choose $n \in \mathbb{N}$ and let β be obtained from α^{n+1} by deleting the last component. By Lemma 3.7, $f(\beta) = f(\alpha^n)$. By B2, $f(\beta \oplus n + 1) > f(\alpha^n \oplus \varnothing)$ if, and only if, $n + 1 > d_\beta \geq \varnothing$. Since $d_\beta = n$, it follows that $f(\alpha^{n+1}) = f(\beta \oplus n + 1) > f(\alpha^n \oplus \varnothing) = f(\alpha^n)$.

Step 2: for all $n \in \mathbb{N}$ and $x \in H_n$, $f(x) = h(x)$. Suppose not: for some $n \in \mathbb{N}$ and $x \in H_n$, $f(x) \neq h(x)$. Case 1: $f(x) < h(x)$. This means that $f(x) \leq n - 1$. By Lemma 3.7, $f(x) = f(\alpha^n)$. By step 1, $f(\alpha^n) > f(\alpha^{n-1}) > \ldots > f(\alpha^0)$. Therefore, $\{f(\alpha^n), f(\alpha^{n-1}), \ldots, f(\alpha^0)\}$ consists of $n + 1$ different non-negative integers whose maximum is $n - 1$ or smaller, which is impossible. Case 2: $f(x) > h(x)$. This means that $f(x) \geq n + 1$. By Lemma 3.7, $f(x) = f(\alpha^n)$. By step 1, $f(\alpha^n) > f(\alpha^{n-1}) > \ldots > f(\alpha^0)$. Hence, $\{f(\alpha^n), f(\alpha^{n-1}), \ldots, f(\alpha^0)\}$ consists of $n + 1$ different non-negative integers whose maximum is at least $n + 1$. As a result, some $k \in \{0, 1, \ldots, n\}$ does not belong to $\{f(\alpha^n), f(\alpha^{n-1}), \ldots, f(\alpha^0)\}$. By B1, some $y \in D$ satisfies $f(y) = k$. Since $k \notin \{f(\alpha^n), f(\alpha^{n-1}), \ldots, f(\alpha^0)\}$, by Lemma 3.7, $y \notin H_0 \cup \ldots \cup H_n$. Given that $f(\alpha^n) > k$, by step 1, there is no $r > n$ with $f(\alpha^r) = k$. And, by Lemma 3.7, there is no $r > n$ and $z \in H_r$ with $f(z) = k$. This contradicts B1.■

Proposition 3.9. An index$_2$ f satisfies B2 and B3 if, and only if, f is the Hirsch index.

Proof. "⇐" Remark 3.1. "⇒" Let f be an index$_2$ satisfying B1 and B2. Step 1: for all $n \in \mathbb{N}$, $f(\alpha^{n+1}) > f(\alpha^n)$. Proved as step 1 in the proof of Proposition 3.8. Step 2: for all $x \in H_0$, $f(x) = h(x)$. By B3, there is $x \in D_0$ such that $f(x) = 0$; that is, $f(\varnothing) = 0 = h(\varnothing)$. By Lemma 3.7, for all $x \in H_0$, $f(x) = f(\varnothing) = 0 = h(x)$. Step 3: for all $n \in \mathbb{N}\backslash\{0\}$ and $x \in H_n$, $f(x) = h(x)$. By step 2, choose $n \in \mathbb{N}\backslash\{0\}$ and, arguing inductively, suppose that, for all $k \in \{0, 1, \ldots, n - 1\}$ and $x \in H_k$, $f(x) = h(x)$. It must be shown that, for all $x \in H_n$, $f(x) = h(x)$. By B3, there is $y \in D_n$ such that $f(y) = n$. By the induction hypothesis, $y \notin H_0 \cup \ldots \cup H_{n-1}$. For $r > n$, by definition of H_r, $D_n \cap H_r = \varnothing$. In view of this, for all $r \in \mathbb{N}$ with $r > n$, $y \notin H_r$. In conclusion, $y \in H_n$. By Lemma 3.7, for all $x \in H_n$, $f(x) = f(y) = n = h(x)$.■

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