

Convexity, subadditivity and the Hirsch index

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Abstract

The Hirsch index is a number that synthesizes a researcher's output. It is defined as the maximum number h such that the researcher has h papers with at least h citations each. Four axiomatic characterizations of the Hirsch index are suggested. The basic characterization requires three axioms: convexity, strict subadditivity and a condition stating that the index of an output with n papers each of which has n citations is n and, for all $n \in \{0, 1\}$, that the index of an output in which every paper has n citations is n . The other three characterizations replace this condition by intuitively more appealing axioms.

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1. Introduction

The Hirsch (2005) index is a measure that summarizes a researcher's output by reducing this output to a number. Suppose that a researcher r is characterized by a set of papers and, for each paper, the number of citations of that paper. Then the Hirsch index of r is the maximum number h of papers by r having at least h citations each. For instance, if x is the output consisting of one paper that is cited ten times and y is the output consisting of ten papers each of which is cited once, then x and y have the same Hirsch index: 1.

Woeginger (2008a, 2008b) and Quesada (2009, 2010) have already suggested characterizations of the Hirsch index. Marchant (2009), instead of characterizing the index itself, characterizes the ranking that the Hirsch index induces on outputs. This paper suggests another four characterizations of the Hirsch index. All four are based on two axioms that have not been previously considered in axiomatizations of the Hirsch index: convexity and strict subadditivity.

Convexity states that if two outputs have the same index, then any output lying between them has also the same index. The characterizations by Woeginger (2008a, 2008b) and Quesada (2009) hinge on the property of monotonicity: more citations or papers do not reduce the index. Convexity is an assumption weaker than monotonicity: monotonicity implies, but is not implied, by convexity.

Strict subadditivity is the property according to which the index corresponding to the aggregation of two outputs is smaller than the sum of the index associated with each output. Subadditivity requires the index of the aggregation of two outputs to be not greater than the sum of the indices of the two outputs. The Hirsch index h satisfies subadditivity but not strict subadditivity: $h(3, 2)$ is not smaller than $h(2, 1) + h(1, 1)$. Nonetheless, strict subadditivity holds on certain domains. It is for one such domain that strict subadditivity is postulated.

The first characterization is based on convexity, strict subadditivity and a third axiom (the normalization condition NOR in Section 2) setting the value of the index for certain types of output. NOR asserts that outputs without any citation deserve index 0; that outputs in which every paper has only one citation deserve index 1; and that outputs with n papers each one of which has n citations deserve index n .

Though NOR does probably not express an appealing property of an index, it is worth noticing that any index based on some convex combination of the number of citations

satisfies NOR. It should also be noticed that an axiomatization may serve at least two purposes: the understanding and the justification of what is being characterized. In this respect, the axiomatization of the Hirsch index that invokes NOR tries to contribute to its understanding: the initial decisions on the evaluation of outputs that NOR represents generate the Hirsch index by just requiring the index to satisfy convexity and (essentially) strict subadditivity.

The other three characterizations are more concerned with the issue of trying to justify the reasonability of the Hirsch index by providing axiomatizations based on axioms that appear justifiable. Convexity seems to be one such axiom. Strict subadditivity may be acceptable as well as expression of a conservative evaluation policy: doing the same twice does not guarantee the duplication of the index. Subadditivity captures the idea that output evaluation is subject to some sort of diseconomies of scale: the index associated with the aggregation of two outputs is never higher than the sum of the indices of the two outputs evaluated independently. This suggests that the effort (measured in citations) necessary to increase the index increases itself with the index.

The second characterization is obtained from the first one by replacing NOR with two axioms whose conjunction imply NOR. These two axioms set upper and lower bounds to the index, which is limited above by the number of citations received by the most cited paper and limited below by the minimum of two values: the number of cited papers and the number of citations of the least cited paper. Both axioms seem acceptable if the index is supposed to be measured in citations and the aim is to determine the number of citations that a representative paper in the output receives.

The third characterization replaces the (perhaps questionable) lower bound condition by a strong monotonicity condition: if output x has more papers than output y and each paper in x is cited more times than the most cited paper in y , then the index attributed to x is higher than the one assigned to y . Insofar as x appears to be superior to y in every respect (more papers and more citations for each paper), this property may be considered a natural requirement to be satisfied by a sensible index.

The last characterization adds three requirements to convexity and strict subadditivity. Two of them are mild: some output has index 1 and outputs without citations have index 0. The third is a constant returns to scale axiom: the k -fold enlargement of the smallest output (according to the lexicographic ordering) having index n is the smallest output having index kn . Hence, rescaling the smallest output having index n by scale k generates the smallest output having index kn .

2. Definitions and main axioms

Let \mathbb{N} be the set of non-negative integers. Define V to be the set of all vectors $x = (x_1, x_2, \dots, x_n)$ such that: (i) $n \in \mathbb{N} \setminus \{0\}$; (ii) $x_1 \geq x_2 \geq \dots \geq x_n$; and (iii) for all $i \in \{1, \dots, n\}$, $x_i \in \mathbb{N}$. The default interpretation is that the member $x = (x_1, \dots, x_n)$ of V represents the research output consisting of n papers such that, for $i \in \{1, \dots, n\}$, paper i receives x_i citations and such that papers are listed from most to least cited.

A more general interpretation of (x_1, \dots, x_n) is that the vector comprises two elements. One is given by the set $\{1, \dots, n\}$, representing research items under the direct control of the researcher: publications, papers in conferences, working papers, patents, doctoral theses supervised, membership of editorial boards, membership of research groups, etc. One of those items may be chosen as a standard of measurement with respect to which the rest are measured. For instance, one element in $\{1, \dots, n\}$ may be equivalent to one publication, or three conference papers, or two doctoral theses supervised. The second element is given by an evaluation, not under the control of the researcher, of each item in $\{1, \dots, n\}$. The value x_i represents the evaluation associated with item i , expressed in units that are common to all the items. Whatever the interpretation adopted, the problem consists of reducing any such vector (x_1, \dots, x_n) to a number.

For $x \in V$: (i) d_x is the number of components (or dimension) of vector x ; (ii) c_x is the number of components of x different from 0 (the number of papers having some citation); (iii) $x^+ = \max\{x_1, \dots, x_n\}$; and (iv) $x^- = \min\{x_1, \dots, x_n\}$. The empty vector, denoted by \emptyset , represents the output consisting of zero papers. When $x = \emptyset$, the convention is that $d_x = c_x = x^+ = x^- = 0$. A researcher's output is represented by a member of the set $X = V \cup \{\emptyset\}$. For $n \in \mathbb{N}$, define $X_n = \{x \in X : d_x = n\}$ to be the set of outputs having exactly n papers.

Definition 2.1. A research output index (or index, for short) is a mapping $f: X \rightarrow \mathbb{N}$.

Though this definition of index is restrictive, it can be justified by the presumption that the units of measurement of the index are papers or citations. For instance, $f(x)$ can be interpreted as the number of valuable papers in output x or as a representative number of citations of the papers in x .

Definition 2.2. The Hirsch index is the index h such that, for all $x \in X$: (i) if $c_x = 0$, then $h(x) = 0$; and (ii) if $c_x \neq 0$, then $h(x) = \max\{n \in \{1, \dots, d_x\} : x_n \geq n\}$.

For $n \in \mathbb{N}$, 0^n is the member of X_n such that $c_x = 0$ and 1^n is the member of X_n in which each of the n papers has only one citation. For $n \in \mathbb{N} \setminus \{0\}$, c^n is the member of X_n in which each of the n papers has n citations.

NOR. *Normalization.* (i) For all $n \in \mathbb{N} \setminus \{0\}$, $f(c^n) = n$. (ii) For all $n \in \mathbb{N}$, $f(0^n) = 0$. (iii) For all $n \in \mathbb{N} \setminus \{0, 1\}$, $f(1^n) = 1$.

NOR can be viewed as a condition that defines the units of measurement of the index. An output in which no paper receives a citation is assigned value 0. An output in which every paper receives only one citation is assigned value 1. And an output with n papers such that each paper receives n citations is assigned value n . Hence, $f(\emptyset) = f(0, \dots, 0) = 0$, $f(1, \dots, 1) = 1$, $f(2, 2) = 2$, $f(3, 3, 3) = 3$, and so on. Although NOR is probably not an example of an appealing axiom, it will prove itself a helpful instrumental axiom: Section 4 suggests axioms that, by implying NOR, can be used instead of NOR in the characterization of the Hirsch index (Proposition 3.3) that relies on NOR.

Let the binary relation \geq on X be such that, for all $x \in X$ and $y \in X$, $x \geq y$ if and only if: (i) $d_x \geq d_y$; and (ii) for all $i \in \{1, \dots, d_y\}$, $x_i \geq y_i$.

CON. *Convexity.* For all $x \in X$, $y \in X$ and $z \in X$, if $x \geq z \geq y$ and $f(x) = f(y)$, then $f(z) = f(y)$.

CON requires that if output z lies between two outputs x and y having the same index, then z must have the same index as x and y . CON can also be viewed as a monotonicity condition: if the changes from y to $x \geq y$ do not alter the index, then smaller changes should also be incapable of modifying the index. CON is weaker than Woeginger's (2008a, p. 225; 2008b, p. 299) monotonicity.

MON. *Monotonicity.* For all $x \in X$ and $y \in X$, $x \geq y$ implies $f(x) \geq f(y)$.

If f satisfies MON, then $x \geq z \geq y$ implies $f(x) \geq f(z) \geq f(y)$. In view of this, $f(x) = f(y)$ implies $f(z) = f(y)$. This proves that MON implies CON. The index f such that, for all $x \in X$, $f(x) = h(x)$ if $h \geq 2$, $f(x) = 1$ if $h(x) = 0$ and $f(x) = 0$ if $h(x) = 1$ satisfies CON, but not MON. As an example when an index can take values on the set of rational numbers, $f(x) = 1 / (1 + h(x))$ satisfies CON, but not MON.

For $x \in X$ and $y \in X$ satisfying $d_x \geq d_y$, let $x + y$ and $y + x$ both designate the member z of X such that: (i) for all $i \in \{1, \dots, d_y\}$, $z_i = x_i + y_i$; and (ii) for all $i \in \{d_y + 1, \dots, d_x\}$,

$z_i = x_i$. For $x \in X \setminus \{\emptyset\}$, define x_{-1} to be the member of X obtained from x by deleting the first component x_1 of x . Formally, for $n \geq 1$ and $x \in X_n$: (i) if $d_x \geq 2$, then x_{-1} is the member y of X_{n-1} such that, for all $i \in \{1, \dots, n-1\}$, $y_i = x_{i+1}$; and (ii) if $d_x = 1$, then $x_{-1} = \emptyset$.

SUB. Strict subadditivity. For all $x \in X \setminus \{\emptyset\}$, if $f(x_{-1}) < f(x)$, then, for all $y \in X$ such that $y^+ \neq 0$, $f(x+y) < f(x) + f(y)$.

An index f is subadditive when $f(x+y) \leq f(x) + f(y)$. SUB selects a domain of X in which the index must be strictly subadditive (with respect to outputs not consisting of papers without citations). Suppose that the most cited paper in output x is decisive: the removal of that paper reduces the index. Let y be an output in which some paper has some citation. Then the index of the enlargement $z = x + y$ of x must be smaller than the sum of the index of x and the index of y .

3. Benchmark result

Remark 3.1. The Hirsch index satisfies NOR, MON, CON, and SUB.

It is not difficult to verify that the Hirsch index satisfies NOR and MON. Since MON implies CON, it also satisfies CON. With respect to SUB, suppose $h(x_{-1}) < h(x)$ and $y^+ \neq 0$. It follows from $h(x_{-1}) < h(x)$ that $x_{h+1} < h(x)$, so $x_{h+1} \leq h(x) - 1$. Case 1: $h(y) \leq h(x)$. This means that paper $h+1$ in $x+y$ receives at most $h(y) + h(x) - 1$ citations. Therefore, $h(x+y) < h(x) + h(y)$. Case 2: $h(y) > h(x)$. In this case, $x+y$ is such that: (i) each of the first $h(x)$ papers has at least $h(x) + h(y)$ citations; (ii) the papers after paper $h(x)$ and up to paper $h(y)$ have at least $h(y)$ citations each; and (iii) the remaining papers have at most $h(x) - 1 + h(y)$ citations each. In view of this, $h(x+y) \leq h(x) + h(y) - 1 < h(x) + h(y)$.

Lemma 3.2. If an index f satisfies NOR and CON, then, for all $x \in X$, $f(x) \geq h(x)$.

Proof. Part 1: for all $x \in X_0 \cup X_1$, $f(x) \geq h(x)$. Let $x \in X_0 \cup X_1$. If $c_x = 0$, then, by NOR(ii), $f(x) = 0 = h(x)$. If $c_x = 1$ and $x^+ = 1$, then, by NOR(iii), $f(x) = 1 = h(x)$. Finally, if $c_x = 1$ and $x^+ > 1$, then suppose $f(x) = k < 1 = h(x)$. This and $f(x) \in \mathbb{N}$ imply $f(x) = 0$. By NOR, $f(0) = 0$ and $f(1) = 1$. Since $x \geq (1) \geq (0)$ and $f(x) = f(0) = 0$, by CON, $f(1) = f(0)$: contradiction. Part 2: for all $x \in X \setminus (X_0 \cup X_1)$, $f(x) \geq h(x)$. Taking Part 1 as the base case of an induction argument, choose $n \in \mathbb{N} \setminus \{0, 1\}$ and suppose that, for all $k \in \{0, 1,$

$\dots, n-1\}$ and $x \in X_k, f(x) \geq h(x)$. To show that, for all $x \in X_n, f(x) \geq h(x)$, pick $x \in X_n$. With $h = h(x)$ and $k = f(x)$, suppose $k < h$. This assumption contradicts the definition of index if $h = 0$, so $x_1 \geq 1$ and $h \geq 1$. By NOR(i), $f(c^h) = h$ and $f(c^k) = k$. Since $x \geq c^h \geq c^k$, it follows from CON and $f(x) = f(c^k) = k$ that $f(c^h) = k$: contradiction. ■

Proposition 3.3. An index f satisfies NOR, CON and SUB if and only if f is the Hirsch index.

Proof. “ \Leftarrow ” Remark 3.1. “ \Rightarrow ” Let index f satisfy NOR, CON and SUB. For $n \in \mathbb{N}$, $C_n = \{x \in X: x^+ = n\}$ is the set of outputs whose most cited paper has n citations. The proof is by induction on the sets C_n .

Part 1: $f = h$ on $C_0 \cup C_1$. By NOR (i), for all $n \in \mathbb{N}$, $f(0^n) = 0 = h(0^n)$. By NOR(i) and NOR(iii), for all $n \in \mathbb{N} \setminus \{0\}$ $f(1^n) = 1 = h(1^n)$. And if $x = (1^k, 0^r)$, then, as shown, $f(1^k) = 1 = f(1^{k+r})$. Since $1^{k+r} \geq x \geq 1^k$, by CON, $f(x) = f(1^k) = 1 = h(x)$.

Part 2: for all $n \in \mathbb{N} \setminus \{0, 1\}$, $f = h$ on C_n . Taking Part 1 as the base case of an induction argument, choose $n \in \mathbb{N} \setminus \{0, 1\}$ and suppose that, for all $k \in \{1, \dots, n-1\}$, $f = h$ on C_k . To show that $f = h$ on C_n , choose $x \in C_n$. Letting $h = h(x)$, by Lemma 3.2, $f(x) \geq h$. Assume $f(x) > h$. The aim is to reach a contradiction. Since $h = h(x)$, $x_h \geq h$. In addition, $x^+ = n$ implies $n \geq h$.

Case 1: $x_h > h$. Let $z \in X$ satisfy: (i) $d_z = d_x$; (ii) for all $i \in \{1, \dots, c_x\}$, $z_i = x_i - 1$; and (iii) for all $i \in \{c_x + 1, \dots, d_x\}$, $z_i = 0$. It follows from $x_h > h$ that $h(z) = h(x)$. If $n = h$, then $h(z_{-1}) < h(z)$. If $n > h$, then $h(x) = h$ implies $x_{h+1} \leq h$. In view of this, paper h in z_{-1} (which is paper $h+1$ in x) has at most $h-1$ citations. Consequently, $h(z_{-1}) \leq h-1 < h = h(z_{-1})$. Summarizing, no matter whether $n = h$ or $n > h$, $h(z_{-1}) < h(z)$. By the induction hypothesis, $f(z_{-1}) = h(z_{-1})$ and $f(z) = h(z)$. Therefore, $f(z_{-1}) < f(z)$. By SUB, $f(x) < f(z) + f(y)$, where $y \in X$ satisfies: (i) $d_y = d_x$; (ii) for all $i \in \{1, \dots, c_x\}$, $y_i = 1$; and (iii) for all $i \in \{c_x + 1, \dots, d_x\}$, $y_i = 0$. By Part 2, $f(y) = 1$. As a result, $f(z) > f(x) - 1$. Given that $f(x) \in \mathbb{N}$, $f(x) > h$ implies $f(x) \geq h+1$. Accordingly, $f(z) > f(x) - 1 \geq (h+1) - 1 = h = h(z)$. This contradicts the induction hypothesis.

Case 2: $x_h = h$. Let $z \in X$ satisfy: (i) $d_z = d_x$; (ii) for all $i \in \{1, \dots, h\}$ such that $x_i = h$, $z_i = h+1$; and (iii) otherwise, $z_i = x_i$. Output z is obtained from x by adding one citation to paper h and, if necessary, to those papers that precede paper h and have the same number of citations as paper h . By Lemma 3.2, $f(z) \geq h(z)$. Since $h(x) = h$ and z is obtained from x by increasing the number of citations of some of the first h papers, $h(z)$

$= h(x) = h$. Consequently, $f(z) \geq h$. Case 2a: $f(z) = h$. By NOR(i), $f(c^h) = h$. Since $z \geq x \geq c^h$ and $f(z) = f(c^h) = h$, by CON, $f(x) = h$: contradiction. Case 2b: $f(z) > h$. In this case, $z_h > h$ and $h(z) = h$. This is the same situation represented by Case 1, which (by replacing x with z) shows how to reach a contradiction from $f(z) > h$. ■

The following examples prove that no axiom in Proposition 3.3 is redundant.

Example 3.4. Let f be the index such that, for all $x \in X$, $f(x) = 0$. Then f satisfies CON and SUB; does not satisfy NOR; and is not the Hirsch index.

Example 3.5. Let f be the index such that, for all $x \in X_0 \cup X_1$, $f(x) = h(x)$ and, for all $x \in X \setminus (X_0 \cup X_1)$, $f(x) = x_2$. Then f satisfies NOR and SUB; does not satisfy CON (since $f(0) = f(3, 0) = 0$ but $f(1) = 1$); and is not the Hirsch index.

Example 3.6. Let f be the index such that, for all $x \in X$, $f(x) = x_1$. Then f satisfies NOR and CON; does not satisfy SUB (if $x = (2, 1)$, then $f(x_{-1}) < f(x)$ but $f(x + x) = 2f(x)$); and is not the Hirsch index.

The following example shows that Proposition 3.3 does not hold if the index can take values in the set of non-negative rational numbers.

Example 3.7. With $X^* = \{x \in X: h(x) = 2 \text{ and } x_1 \geq 3\}$, let f be the mapping defined on X such that, for all $x \in X^*$, $f(x) = 2.1$ and, for all $x \in X \setminus X^*$, $f(x) = h(x)$. Then f is not the Hirsch index but satisfies NOR, CON and SUB.

4. Additional results

Since there is no obvious reason to impose NOR on an index, Proposition 3.3 cannot be presented as a justification of the Hirsch index. Instead, it may contribute to understand it better: the evaluation decisions captured by NOR generate the Hirsch index by just adding convexity (a reasonable continuity requirement) and an appropriate amount of strict subadditivity (an assumption that is justifiable as expression of a conservative output evaluation policy). The results in this section are motivated by the idea of replacing NOR by axioms that, implying NOR, may be considered more appealing than NOR.

UPB. *Upper boundedness.* For all $x \in X$, $f(x) \leq x^+$.

LOB. *Lower boundedness.* For all $x \in X$, $f(x) \geq \min\{x^-, c_x\}$.

UPB states that the index of output x is bounded above by the number of citations of the most cited paper in x . LOB establishes a lower bound equal to the smallest of two values: the number of citations of the least cited paper and the number of cited papers. UPB and LOB appear to be reasonable properties of an index that is assumed to associate, with each output, a representative number of citations in the output.

Remark 4.1. UPB and LOB imply NOR.

Proposition 4.2. An index f satisfies UPB, LOB, CON and SUB if and only if f is the Hirsch index.

Proof. Proposition 3.3, Remark 4.1 and the fact that the Hirsch index satisfies UPB and LOB. ■

Let the binary relation $>$ on X be such that, for all $x \in X$ and $y \in X$, $x > y$ if and only if: (i) $d_x > d_y$; and (ii) for all $i \in \{1, \dots, d_x\}$, $x_i > y^+$. For instance, $f(3, 2) > f(2)$ but neither $f(3, 2) > f(2, 1)$ nor $f(0, 0) > f(\emptyset)$ holds. Hence, $x > y$ if and only if x has more papers than y and each paper in x has more citations than the most cited paper in y .

SMO. *Strong monotonicity.* For all $x \in X$ and $y \in X$, $x > y$ implies $f(x) > f(y)$.

SMO indicates what is sufficient for the index to increase: if more papers are added to output x to generate another output y and all the papers in y receive more citations than the most cited paper in x , then the index of the new output y should be higher than the index of the initial output x .

Remark 4.3. UPB, SMO and CON imply NOR.

If $x^+ = 0$, then, by definition of index, $f(x) \geq 0$ and, by UPB, $f(x) \leq x^+ = 0$. This proves NOR(ii). Letting $x^0 = \emptyset$ and taking $f(\emptyset) = 0$ as the base case of an induction argument, choose $n \in \mathbb{N} \setminus \{0\}$ and suppose that, for all $k \in \{0, 1, \dots, n-1\}$, $f(c^k) = k$. In particular, $f(c^{n-1}) = n-1$. By SMO, $f(c^n) > f(c^{n-1}) = n-1$. By UPB, $f(c^n) \leq n$. Since $f(c^n) \in \mathbb{N}$, $f(c^n) > n-1$ and $f(c^n) \leq n$ imply $f(c^n) = n$. This proves NOR(i). With respect to NOR(iii), by SMO, $f(1) > f(\emptyset)$. As shown, $f(\emptyset) = 0$. By UPB, $f(1) \leq 1$. Since $f(1) \in \mathbb{N}$, $f(1) = 1$. Choose $n \in \mathbb{N} \setminus \{0, 1\}$ and suppose $f(1^n) \neq 1$. By UPB and the fact that $f(1^n) \in \mathbb{N}$, $f(1^n) = 0$. By CON, $f(\emptyset) = f(1^n) = 0$ and $1^n \geq (1) \geq \emptyset$ imply $f(1) = f(\emptyset) = 0$: contradiction.

Proposition 4.4. An index f satisfies UPB, SMO, CON and SUB if and only if f is the Hirsch index.

Proof. Proposition 3.3, Remark 4.3 and the fact that the Hirsch index satisfies UPB and SMO. ■

The following conventions are adopted: (i) for all $n \in \mathbb{N}$, $\emptyset < n$; and (ii) for all $x \in X$ and $n > d_x$, $x_n = \emptyset$. The lexicographic ordering on X is the binary relation $<_L$ on X such that, for all $x \in X$ and $y \in X \setminus \{x\}$, $x <_L y$ if and only if, for some $i \in \mathbb{N}$: (i) $x_i < y_i$; and (ii) if $i > 0$, then, for all $k < i$, $x_k = y_k$. For instance, $\emptyset <_L (0) <_L (0, 0)$, $(2, 1) <_L (2, 2)$, $(2, 2, 2) <_L (3, 1)$, and $(1, 1) <_L (1, 1, 0)$. Observe that, for all $x \in X$ and $y \in X \setminus \{x\}$, either $x <_L y$ or $y <_L x$.

For $n \in \mathbb{N}$, define s^n to be the member of X , if it exists, such that $f(s^n) = n$ and, for all $x \in X$ satisfying $f(x) = n$, $s^n <_L x$. In words, s^n is the smallest output, according to the lexicographic ordering, having index n . For $x \in X \setminus \{\emptyset\}$ and $k \in \mathbb{N} \setminus \{0\}$, define $k \cdot x$ to be the member y of X such that: (i) $d_y = kd_x$; (ii) for all $i \in \{1, \dots, d_x\}$, $y_i = kx_i$; and (iii) for all $i \in \{d_x + 1, \dots, d_y\}$, $y_i = kx^-$. For $x = \emptyset$ and $k \in \mathbb{N} \setminus \{0\}$, define $k \cdot x = x$. Loosely speaking, $k \cdot x$ is the output obtained from x by rescaling x by k , that is, by multiplying papers and citations by the scale k .

CRE. *Constant returns to scale.* For all $n \in \mathbb{N} \setminus \{0\}$ and $k \in \mathbb{N} \setminus \{0\}$, $k \cdot s^n = s^{nk}$.

CRE holds that if the smallest output necessary to obtain index $n \neq 0$ is rescaled by k , then the result is the smallest output necessary to obtain index kn . This suggests that the index, at least for this specific case, is subject to constant returns to scale: a k -fold enlargement of that output generates a k -fold enlargement of the index.

NOC. *No citation, zero index.* For all $n \in \mathbb{N} \setminus \{0\}$, $f(0^n) = 0$.

ONE. *Existence of an output with index one.* There is $x \in X$ such that $f(x) = 1$.

Lemma 4.5. ONE, NOC, CRE, CON and SUB imply NOR.

Proof. Part 1: $f(\emptyset) = 0$. Suppose not: $f(\emptyset) = r \neq 0$. By definition of index, $r > 0$. In addition, for all $x \in X \setminus \{\emptyset\}$, $\emptyset <_L x$. Therefore, $\emptyset = s^r$. By CRE, $2 \cdot \emptyset = s^{2r}$. That is, $f(2 \cdot \emptyset) = 2r$. But $2 \cdot \emptyset = \emptyset$, so $f(2 \cdot \emptyset) = f(\emptyset) = r$: contradiction.

Part 2: $f(1) \neq 0$. Suppose not: $f(1) = 0$. By ONE, let $s^1 = x$. Obviously, $x \neq (1)$. By NOC, $c_x \geq 1$. Consider x_{-1} . If $f(x_{-1}) = f(x)$, then, since $x_{-1} <_L x$, a contradiction of the assumption that $s^1 = x$ is reached. If $f(x_{-1}) > f(x)$, then there is $k \in \mathbb{N} \setminus \{0, 1\}$ such that $f(x_{-1}) = k$. By CON, $s^1 = x$ implies $k \cdot x = s^k$. But $x_{-1} <_L k \cdot x$, which contradicts $k \cdot x = s^k$. If $f(x_{-1}) < f(x)$, then, by SUB, $f(x + (1)) < f(x) + f(1) = f(x) = 1$. Consequently, $f(x + (1)) = 0$. By CON and Part 1, $f(x + (1)) = f(\emptyset) = 0$ and $x + (1) \geq x \geq \emptyset$ imply $f(x) = 0$: contradiction.

Part 3: $f(1) = 1$. Suppose not: $f(1) \neq 1$. By Part 2, $f(1) = r > 1$. By Part 1 and NOC, for all $y \in X$ such that $y <_L (1)$, $f(y) \neq f(1)$. Consequently, $s^r = (1)$. By ONE, let $s^1 = x$. Hence, $(1) <_L x$. By CRE, $s^r = (1)$ and $s^1 = x$ imply $(1) = r \cdot x$. But $(1) <_L x$ yields $(1) <_L r \cdot x$: contradiction.

Part 4: NOR holds. By Part 3, $f(1) = 1$. This and CRE imply NOR(i). NOR(ii) follows from Part 1 and NOC. With respect to NOR(iii), choose $n \in \mathbb{N} \setminus \{0, 1\}$ and suppose that $f(1^n) = r \neq 1$. By Part 3, $n \geq 2$. Case 1: $r = 0$. By Part 1, $f(\emptyset) = 0$. By CON, $(1^n) \geq (1) \geq \emptyset$ and $f(1^n) = f(\emptyset) = 0$ imply $f(1) = 0$, which contradicts Part 2. Case 2: $r > 1$. Since $f(1) = 1$, by CRE, $c^r = s^r$. Accordingly, $(1^n) <_L s^r$, which contradicts the fact that $c^r = s^r$. ■

Proposition 4.6. An index f satisfies ONE, NOC, CRE, CON and SUB if and only if f is the Hirsch index.

Proof. It is not difficult to verify that the Hirsch index satisfies ONE, NOC and CRE (in particular, the Hirsch index is such that, for all $n \in \mathbb{N} \setminus \{0\}$, $s^n = c^n$). In view of this, the result follows from Proposition 3.3 and Lemma 4.5. ■

Proof. The Hirsch index is such that, for all $n \in \mathbb{N} \setminus \{0\}$, $s^n = c^n$. Hence, the Hirsch index satisfies CRE. It also satisfies ONE and NOC, so the result follows from Proposition 3.3 and Lemma 4.5. ■

All the characterizations show the Hirsch index to be the result of a sort of compromise between expansive axioms (which establish when the index raises) and restrictive axioms (which limit the potential increase of the index). In Proposition 4.6, CRE may be viewed as an expansive axiom and the rest of axioms as restrictive. In Proposition 4.4, the expansive axiom is SMO, whereas the restrictive axioms are UPB, CON and SUB. In Proposition 4.2, LOB counts as expansive and UPB, CON and SUB as

restrictive. And in Proposition 3.3, the expansive side is represented by NOR, with CON and SUB defining the restrictive side.

Definition 4.7. The number of papers index is the index p such that, for all $x \in X$, $p(x) = d_x$.

Both ONE and NOC are necessary assumptions in Proposition 4.6. Index p satisfies ONE, CRE, CON and SUB, does not satisfy NOC, and is not the Hirsch index. On the other hand, the index such that, for all $x \in X$, $f(x) = 0$ satisfies NOC, CRE, CON and SUB, does not satisfy ONE, and is not the Hirsch index.

It is worth noticing that p satisfies both CRE and CON and that it also satisfies SUB except when $x = \emptyset$. On the one hand, for all $x \in X \setminus \{\emptyset\}$, $p(x_{-1}) < p(x)$ is always the case. And, on the other hand, for all $x \in X \setminus \{\emptyset\}$ and $y \in X$ such that $y^+ \neq 0$, $p(x + y) = \max\{d_x, d_y\} < d_x + d_y = p(x) + p(y)$. Inspection of the proofs of Lemma 4.5 and Proposition 3.3 makes clear that the case $x = \emptyset$ may be excluded from the definition of SUB, so p would fail to satisfy only one of the five axioms that would characterize the Hirsch index in Proposition 4.6: NOC. The characterization of p suggested next reinforces the somewhat curious impression that, from an axiomatic point of view, indices h and p appear to be closely related despite the fact that p completely disregards one of the dimensions of outputs: citations.

DIM. *Dimension is an upper bound.* For all $x \in X$, $f(x) \leq d_x$.

DIM points to the interpretation of an index as providing the number of quality papers in an output. The Hirsch index satisfies DIM. Given Proposition 4.8 next, the only axiom that the Hirsch index fails to satisfy among those characterizing the number of papers index is the axiom setting the unit of measurement: $f(0) = 1$.

Proposition 4.8. An index f satisfies DIM, CRE, CON and $f(0) = 1$ if and only if f is the number of papers index.

Proof. “ \Leftarrow ” It is evident that $p(0) = 1$ and that p satisfies DIM. Letting $(0^0) = \emptyset$, it is not difficult to verify that, for all $n \in \mathbb{N}$, $s^n = (0^n)$. Since $0^n = n \cdot (0)$, p satisfies CRE. With respect to CON, let $x \geq z \geq y$ and $p(x) = p(y)$. Hence, $d_x = d_y$. It then follows from $x \geq z$ that $d_x \geq d_z$ and from $z \geq y$ that $d_z \geq d_y$. As a result, $d_z = d_y$ and, consequently, $p(z) = p(y)$. “ \Rightarrow ” Part 1: $f(\emptyset) = 0 = p(\emptyset)$. This follows from DIM and the fact that $f(\emptyset) \in \mathbb{N}$. Part 2: $f(0, \dots, 0) = p(0, \dots, 0)$. Given that \emptyset is the only member x of X such that $x <_L (0)$,

$f(\emptyset) = 0$ and $f(0) = 1$ imply $(0) = s^1$. By CRE, for all $n \in \mathbb{N} \setminus \{0\}$, $s^n = n \cdot (0)$. Since $n \cdot (0) = (0^n)$, $s^n = (0^n)$. This means that, for all $n \in \mathbb{N} \setminus \{0\}$, $f(0^n) = n = p(0^n)$. Part 3: for all $x \in X$ such that $\emptyset \neq x \neq (0, \dots, 0)$, $f(x) = p(x)$. Suppose not: $f(x) = r \neq p(x)$. By DIM, $r < d_x$. Let $n = d_x$. Case 1: $r = 0$. By Part 1, $f(\emptyset) = 0$. This, CON and $x \geq (0) \geq \emptyset$ imply $f(0) = f(\emptyset) = 0$, which contradicts $f(0) = 1$. Case 2: $r \geq 1$. By Part 2, $f(0^n) = n > r = f(0^r)$. By CON, $f(x) = f(0^r) = r$ and $x \geq 0^n \geq 0^r$ imply $f(0^n) = f(0^r) = r$: contradiction. ■

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