Up/downward preference aggregation

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Abstract

The existence of a dictator for the aggregation of strict preferences over a given set A of m alternatives is shown when simpler versions of the following downward and upward procedures are simultaneously used and forced to deliver a social welfare function. Choice functions $d_1, \dots, d_m, u_1, \dots, u_m$ mapping profiles of alternatives into alternatives are arbitrarily selected. For preference profile P , the top x_1 in the collective preference p associated with P is the value that d_1 associates with the profile of the top-ranked alternatives in P . The second most preferred alternative x_2 in p is the value that d_2 associates with the profile of the top-ranked alternatives in P restricted to $A \setminus \{x_1\}$. The third alternative in p is the value that d_3 associates with the profile of the top-ranked alternatives in P restricted to $A \setminus \{x_1, x_2\}$. And so on until the m positions are filled. The upward procedure resorts to $u_1, \dots, u_m$ in a similar way but starting from the bottom-ranked alternatives.

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1. Introduction

If there is one, the representative economic model has a set of states of the world, agents making decisions at every state and rules establishing how the decisions are made and how such decisions determine the passage from one state to another. The analysis of this type of model unfolds in two directions: one trying to characterize the states that can be reached under the given rules; the other, dealing with the evaluation of the collective desirability of such states. This evaluation relies on the selection of criteria determining whether a state is, from a collective point of view, more desirable than another one.

Arrow's [1, p. 97] theorem pointed to the possibility that making progress in the second direction could not be an easy task. This theorem is formulated in a framework in which the aim is to define an aggregation rule that obtains a collective preference over states exclusively from the agents' preferences over states. Such aggregation rules could next be used to evaluate states from a collective point of view and, in this way, complete the analysis of the representative economic model along the second direction.

In the version of Arrow's theorem for preferences in which indifference is not present and all preferences are allowed, two criteria derived from the majority principle are imposed on preference aggregation rules. One is the Pareto principle: if all the agents prefer state x to state y then x should be preferred to y in the preference associated with the collective of agents. The second is the independence of irrelevant alternatives (IIA) condition: the collective preference between two states only depends on the agents' preferences over those states. The result is that all the aggregation rules satisfying the Pareto principle and IIA are dictatorial: for some agent i , the collective preference always coincides with i 's preference. As a consequence, imposing IIA and the Pareto principle on aggregation rules amounts to letting one agent always determine whether one state is more desirable than another state from a collective point of view.

Arrow's theorem has been shown to be a relatively robust result; see, for instance, [3]. Despite this, recent contributions seem to indicate that the difficulties associated with Arrow's theorem can be overcome or are, at least, in need of reassessment. For example, Campbell and Kelly [2, 4] suggest small departures from IIA that allow the existence of aggregation rules that are far from dictatorial; Saari [6, 7] questions the appropriateness of IIA as a reasonable requirement for the type of aggregation rules considered in Arrow's theorem; and Tangian [8, 9] suggests an interpretation of dictators as representatives that even justifies sustaining a positive assessment of Arrow's theorem.

Arrow's theorem follows a vertical approach to preference aggregation in the sense that IIA and the Pareto principle are conditions stating whether a state is or not above another state in the collective preference ranking. Houy [5] considers an alternative horizontal approach according to which the state that occupies any given position r in the collective preference only depends on the states that occupy position r in the agents' preferences. Specifically, a choice function c_r is associated with each position r in the collective preference, so that the state in position r is the value that c_r attributes to the profile of states occupying position r in the agents' preferences. Houy [5, p. 343] shows that aggregation rules obtained using choice functions to determine each position in the collective preference always output the same preference or has an individual who determines the collective preference for some renaming of states.

Both approaches reduce preference aggregation to simpler aggregation problems. In Arrow's case, preference aggregation over m alternatives is reduced to preference aggregation over just two alternatives. In Houy's case, preference aggregation over m alternatives is reduced to m problems of choosing over alternatives. This paper considers the aggregation of strict preferences following a rather mixed approach.

As in Houy's approach, the collectively most preferred state is assumed to depend only on the profile of most preferred states. So a choice function determines the top state in the collective preference. That choice made, the selected state is removed from the agents' preferences and the same strategy is followed: another choice function determines the top state in the new problem (which is the second state in the general aggregation problem). Therefore, the second most preferred state in the collective preference does not depend on the profile of second most preferred states in the agents' preferences but on the profile of most preferred states when the collectively most preferred state is removed. And so on until all the positions in the collective preference are filled. It is worth noticing that this procedure generates more aggregation rules than IIA (when slightly complemented as in [10]) or Houy's procedure; see Example 4.5.

Two variations on this procedure are considered. First, since there is no reason why aggregation should proceed from top to bottom, a symmetry requirement seems reasonable: the procedure applied from top to bottom should yield the same result as when applied (with possibly different choice functions) from bottom to top. In fact, much less will be assumed, namely, that each procedure applies up to a certain position. For instance, with m positions, the up-down procedure may be asked to provide the states filling positions 1 to r and the down-up procedure to fill positions m to $r + 1$.

And second, both procedures are relaxed in the spirit of IIA. By way of illustration, if the up-down procedure is supposed to operate from positions 1 to r , the weakening consists of requiring that, for each position s from 1 to r , if two preference profiles agree from position 1 to s then the same states that appear in the first s positions in the collective preference associated with one profile must appear (not necessarily in the same order) in the collective preference associated with the other profile. An analogous weakening is applied to the down-up procedure.

Definitions are brought together in Section 2, assumptions in Section 3, results in Section 4 and the proofs of the results in Section 5. The main result, Proposition 4.1, states that, for at least two agents and three states, the only aggregation rules satisfying a mild unanimity condition and whose output can be obtained by a combination of up-down and down-up procedures are the dictatorial ones. A second result, Proposition 4.7, relaxes the unanimity condition by replacing it with an ontoneess property: every state has a preference profile for which it is top-ranked in the collective preference. These two results, combined with Houy's theorem and the family of Arrovian impossibilities, contribute to reinforce the link between independence type assumptions for preference aggregation and the existence of dictators.

2. Definitions

Let $N = \{1, \dots, n\}$ be a finite non-empty set of natural numbers whose n members represent individuals (or voters or agents) and let A be a finite non-empty set whose m elements represent alternatives (or candidates or states).

Definition 2.1. A preference on A is a sequence $(x_1, \dots, x_m)$ of members of A such that $A = \{x_1, \dots, x_m\}$. The set of preferences on A is denoted by L and the set of preference profiles (the set of ways of assigning preferences to individuals) by L^n .

Preferences on A are strict: no member of A is indifferent to any other member of A . The interpretation is then that, in preference $(x_1, \dots, x_m)$, x_s is preferred to x_t if, and only if, $s < t$. For preference $p \in L$, $r \in \{1, \dots, m\}$ and $B \subseteq A$: (i) ${}^r p$ designates the r th element in the sequence p , so ${}^r p$ is the r th most preferred member of A according to p ; and (ii) $p|_B$ denotes the restriction of p to B , so x is preferred to y in $p|_B$ if, and only if, x is preferred to y in p .

For preference $p = (x_1, x_2, \dots, x_{m-1}, x_m) \in L$, $p^* = (x_m, x_{m-1}, \dots, x_2, x_1)$ is the preference obtained from p by reversing p . For $p \in L$, $r \in \{1, \dots, m\}$ and $B \subseteq A$, ${}^r p^*$ designates the r th element in the inverse preference p^* , whereas ${}^1 p^*|_B$ designates the first element in the inverse preference p^* when p^* is restricted to the subset B of A . For $P \in L^n$ and $i \in N$, P_i designates the preference corresponding to individual i in preference profile P .

Definition 2.3. A social welfare function f is a mapping $f: L^n \rightarrow L$.

A social welfare function maps preference profiles (representing the individuals' preferences on A) into preferences (representing a collective preference on A).

Definition 2.4. A social welfare function f is dictatorial if there is $i \in N$ such that, for all $P \in L^n$, $f(P) = P_i$.

3. Assumptions

UNA. For all $P \in L^n$, if ${}^1 P_1 = \dots = {}^1 P_n$ and ${}^m P_1 = \dots = {}^m P_n$ then ${}^1 f(P) = {}^1 P_1$ or ${}^m f(P) = {}^m P_1$.

UNA is the weak unanimity requirement stating that, for each preference profile in which all agents agree in which are the best and worst alternatives, the social welfare function must respect at least of these two agreements.

DOWN _{r} . For every $t \in \{1, \dots, r\}$, there is a mapping $d_t: A^n \rightarrow A$ such that, for all $P \in L^n$: (i) ${}^1 f(P) = d_1({}^1 P_1, \dots, {}^1 P_n)$; and (ii) if $r \geq 2$ then, for all $t \in \{2, \dots, r\}$, ${}^t f(P) = d_t({}^1 P_1|_{A_t}, \dots, {}^1 P_n|_{A_t})$, where $A_t = A \setminus \{{}^1 f(P), \dots, {}^{t-1} f(P)\}$.

DOWN _{r} expresses the postulate that the social welfare function is downward generated, from position 1 to $r \leq m$, by r choice mappings $d_1, \dots, d_r$ as follows: (i) d_1 determines the most preferred alternative in the collective preference using the individuals' most preferred alternatives as inputs; and (ii) for $r \geq 2$, d_r determines the r th most preferred alternative using as inputs the individuals' most preferred alternatives when the alternatives occupying the $r - 1$ first positions in the collective preference are removed.

UP _{r} . For every $t \in \{1, \dots, r\}$, there is a mapping $u_t: A^n \rightarrow A$ such that, for all $P \in L^n$: (i) ${}^1 f(P)^* = u_1({}^1 P_1^*, \dots, {}^1 P_n^*)$; and (ii) if $r \geq 2$ then, for all $t \in \{2, \dots, r\}$, ${}^t f(P)^* = u_t({}^1 P_1^*|_{A_t}, \dots, {}^1 P_n^*|_{A_t})$, where $A_t = A \setminus \{{}^1 f(P)^*, \dots, {}^{t-1} f(P)^*\}$.

UP_r is the symmetric version of $DOWN_r$: whereas the latter starts from position 1 and generates the collective preference downwards, the former starts from the last position m and generates the collective preference upwards. Observe that if $f: L^n \rightarrow L$ satisfies UP_r then $g: L^n \rightarrow L$ such that, for all $P \in L^n$, $g(P_1^*, \dots, P_n^*) = f(P)^*$, satisfies $DOWN_r$.

$IND_r^\downarrow$. For all $P \in L^n$ and $Q \in L^n$, if, for all $t \in \{1, \dots, r\}$ and $i \in N$, ${}^tP_i = {}^tQ_i$ then $\{{}^1f(P), {}^2f(P), \dots, {}^rf(P)\} = \{{}^1f(Q), {}^2f(Q), \dots, {}^rf(Q)\}$.

$IND_r^\downarrow$ is an independence type condition version of $DOWN_r$. By $IND_r^\downarrow$, the following holds for every position $t \in \{1, \dots, r\}$. Consider two preference profiles such that, for every individual i , i 's preference from position 1 to position t is the same in both profiles. Then, in the two collective preferences corresponding to the two profiles, the set of alternatives that occupy the first t positions is the same.

$IND_r^\uparrow$. For all $P \in L^n$ and $Q \in L^n$, if, for all $t \in \{1, \dots, r\}$ and $i \in N$, ${}^tP_i^* = {}^tQ_i^*$ then $\{{}^1f(P)^*, {}^2f(P)^*, \dots, {}^rf(P)^*\} = \{{}^1f(Q)^*, {}^2f(Q)^*, \dots, {}^rf(Q)^*\}$.

$IND_r^\uparrow$ is an independence type condition version of UP_r . It expresses the same requirement as $IND_r^\downarrow$ but considering positions $m, m-1, \dots, m-r+1$ instead of positions $1, 2, \dots, r$. $IND_r^\uparrow$ is just $IND_r^\downarrow$ when all the involved preferences are turned upside down.

$ONTO_r$. For every $x \in A$, there is $P \in L^n$ such that ${}^rf(P) = x$.

$ONTO_r$ attributes each alternative the right of being top-ranked in the collective preference associated with some profile. It can be viewed as a principle of non-discrimination for alternatives.

4. Results

Proposition 4.1. With $n \geq 2 < m$, let $f: L^n \rightarrow L$ be a social welfare function. Then, for all $r \in \{1, \dots, m-2\}$, f satisfies UNA , $IND_r^\downarrow$ and $IND_{m-r-1}^\uparrow$ if, and only if, f is dictatorial.

Remark 4.2. Proposition 4.1 does not hold for $n \geq m = 2$, nor even with the stronger $DOWN_m$ and UP_m replacing $IND_r^\downarrow$ and $IND_{m-r-1}^\uparrow$; see Example 4.3.

Example 4.3. With $n \geq 2$ and $A = \{x, y\}$, let $f : L^n \rightarrow L$ be the non-dictatorial social welfare function that outputs the social preference in which y is preferred to x if, and only if, all individuals prefer y to x . Then UP_2 holds with mappings u_1 and $u_2 = u_1$ such that, for all $\xi \in A^n$, $u_1(\xi) = x$ if, and only if, $\xi = (x^N)$. And $DOWN_2$ holds with mappings d_1 and $d_2 = d_1$ such that, for all $\xi \in A^n$, $d_1(\xi) = y$ if, and only if, $\xi = (y^N)$.

For the simplest case $n = 2$ and $m = 3$, Proposition 4.1 states that f is dictatorial if, and only if, f satisfies UNA, UP_1 and $DOWN_1$. Examples 4.4, 4.5 and 4.6 show that, for this case, no assumption in Proposition 4.1 is redundant.

Example 4.4. With $N = \{1, 2\}$ and $A = \{x, y, z\}$, let $f : L^2 \rightarrow L$ be the non-dictatorial social welfare function such that, for all $P \in L^n$: (i) $^1f(P) = z$; and (ii) $^2f(P) = y$ and $^3f(P) = x$ except when $^3P_1 = ^3P_2 = y$, in which case $^2f(P) = x$ and $^3f(P) = y$. Whereas f satisfies UP_1 and $DOWN_1$, it fails to satisfy UNA.

Example 4.5. With $N = \{1, 2\}$ and $A = \{x, y, z\}$, let $f : L^2 \rightarrow L$ be the non-dictatorial social welfare function such that, for all $P \in L^n$, $^1f(P) = ^1P_1$ and $^2f(P) = ^1P_2|_{A_P}$, where $A_P = A \setminus \{^1P_1\}$. Since 1 is a dictator for position 1, UNA holds. $DOWN_3$ holds with mappings d_1, d_2 and d_3 such that, for all $\xi \in A^2$, $d_1(\xi) = \xi_1$ and $d_2(\xi) = d_3(\xi) = \xi_2$. To see that UP_1 does not hold, let $P \in L^n$ and $Q \in L^n$ satisfy $^1P = ^2Q = (x, y)$, $^2P = (y, z)$, $^3P = ^3Q = (z, x)$ and $^1Q = (y, z)$. Since $^3P = ^3Q$, UP_1 would imply $^3f(P) = ^3f(Q)$, which is not the case: $^3f(P) = z$ and $^3f(Q) = x$.

Example 4.6. With $N = \{1, 2\}$ and $A = \{x, y, z\}$, let $f : L^2 \rightarrow L$ be the non-dictatorial social welfare function such that, for all $P \in L^n$, $^3f(P) = ^3P_1$ and $^2f(P) = ^2P_2|_{A_P}$, where $A_P = A \setminus \{^3P_1\}$. Since 1 is a dictator for position $m = 3$, UNA holds. UP_3 holds with mappings u_1, u_2 and u_3 such that, for all $\xi \in A^2$, $u_1(\xi) = \xi_1$ and $u_2(\xi) = u_3(\xi) = \xi_2$. To see that $DOWN_1$ does not hold, let $P \in L^n$ and $Q \in L^n$ satisfy: $^1P = ^1Q = (z, x)$, $^2P = (y, z)$, $^3P = ^2Q = (x, y)$ and $^3Q = (y, z)$. Since $^1P = ^1Q$, $DOWN_1$ would imply $^1f(P) = ^1f(Q)$, which is not the case: $^1f(P) = z$ and $^1f(Q) = x$.

UNA can be removed from Proposition 4.1 if it is assumed instead that, for instance, the mapping d_1 associated with $IND_1^\downarrow$ satisfies, for all $x \in A$, $d_1(x^N) = x$. Proposition 4.7 replaces UNA with a weaker condition $ONTO_1$ without having to impose any additional property on the mappings associated with the UP and DOWN conditions.

Proposition 4.7. With $n \geq 2 < m$, social welfare function $f : L^n \rightarrow L$ satisfies $ONTO_1$, UP_1 and $DOWN_{m-1}$ if, and only if, f is dictatorial.

Let $\pi = (\pi_1, \dots, \pi_m)$ be a sequence of mappings such that, for all $r \in \{1, \dots, m\}$, $\pi_r : A^n \rightarrow A$. A sequence π is a generating sequence for social welfare function $f : L^n \rightarrow L$ if, for all $P \in L^n$ and $r \in \{1, \dots, m\}$, ${}^r f(P) = \pi_r({}^r P_1, \dots, {}^r P_n)$. Houy's [5, p. 343] Theorem 1 asserts that if a generating sequence exists for a social welfare function $f : L^n \rightarrow L$ then f is either imposed or perm-dictatorial, that is, either (i) there is $p \in L$ such that, for all $P \in L^n$, $f(P) = p$ or (ii) there are $i \in N$ and a bijection $\sigma : A \rightarrow A$ such that, for all $P \in L^n$ and $r \in \{1, \dots, m\}$, ${}^r f(P) = \sigma({}^r P_i)$. The social welfare function f in Example 4.4 is neither imposed nor perm-dictatorial, despite the fact that f satisfies, for $r = 1$, $\text{IND}_r^\downarrow$ and $\text{IND}_{m-r-1}^\uparrow$. Since DOWN_1 and UP_1 are, respectively, equivalent to $\text{IND}_1^\downarrow$ and $\text{IND}_1^\uparrow$, Example 4.4 shows that, for $r \geq 1$ and $m - r - 1 \geq 1$, DOWN_r and UP_{m-r-1} do not guarantee the existence of a generating sequence for f .

Exercise 4.8. For $n \geq 2 < m$, let a social welfare function $f : L^n \rightarrow L$ satisfy DOWN_m and UP_m . Is there a generating sequence for f ?

5. Proofs

For $\xi \in A^n$ and $i \in N$, ξ_i denotes the i th element in the n -tuple ξ . For $x \in A$, $y \in A$ and non-empty $I \subset N$, (x^I, y^{-I}) designates the member ξ of A^n such that: (i) for all $i \in I$, $\xi_i = x$; and (ii) for all $i \in N \setminus I$, $\xi_i = y$. When $I = \{i\}$, (x^i, y^{-i}) stands for $(x^{(i)}, y^{-(i)})$. Similarly, for $j \in N$, $(x^I, y^j, z^{-(I \cup \{j\})})$ designates the member ξ of A^n such that: (i) for all $i \in I$, $\xi_i = x$; (ii) $\xi_j = y$; and (iii) for all $i \in N \setminus (I \cup \{j\})$, $\xi_i = z$. Finally, (x^N) stands for the member ξ of A^n such that, for all $i \in I$, $\xi_i = x$. For $P \in L^n$ and $r \in \{1, \dots, m\}$, ${}^r P$ denotes the member $({}^r P_1, \dots, {}^r P_n)$ of A^n representing the profile of alternatives that occupy position r in preference profile P .

Lemma 5.1. With $n \geq 2 < m$, if $f : L^n \rightarrow L$ satisfies UNA, UP_1 (with associated mapping u_1) and DOWN_1 (with associated mapping d_1) then:

- (i) for all $\xi \in A^n$, $u_1(\xi) \in \{\xi_1, \dots, \xi_n\}$ and $d_1(\xi) \in \{\xi_1, \dots, \xi_n\}$; and
- (ii) there are $i \in N$, $x \in A$ and $y \in A \setminus \{x\}$ such that $d_1(x^i, y^{-i}) = x$.

Proof. Step 1: for all $x \in A$, $u_1(x^N) = x$. Let $x \in A$ and suppose $u_1(x^N) = y \in A \setminus \{x\}$. Choose $P \in L^n$ with ${}^1 P = (y^N)$ and ${}^m P = (x^N)$. By UP_1 , ${}^m f(P) = u_1({}^m P) = u_1(x^N) = y$. Thus, ${}^1 f(P) \neq y$ and ${}^m f(P) \neq x$, contradicting UNA. Step 2: for all $x \in A$, $d_1(x^N) = x$. Let $x \in A$

and suppose $d_1(x^N) = y \in A \setminus \{x\}$. Let $P \in L^n$ satisfy ${}^1P = (x^N)$ and ${}^mP = (y^N)$. By DOWN_1 , ${}^1f(P) = d_1({}^1P) = d_1(x^N) = y$. Hence, ${}^mf(P) \neq y$ and ${}^1f(P) \neq x$, contradicting UNA.

Step 3: for all $\xi \in A^n$, $u_1(\xi) \in \{\xi_1, \dots, \xi_n\}$. Suppose not: for some $x \in A$ and $\xi \in A^n$, $u_1(\xi) = x \notin \{\xi_1, \dots, \xi_n\}$. Choose $P \in L^n$ such that ${}^1P = (x^N)$ and ${}^mP = \xi$. By UP_1 , ${}^mf(P) = u_1({}^mP) = u_1(\xi) = x$. By DOWN_1 and step 2, ${}^1f(P) = d_1({}^1P) = d_1(x^N) = x$: contradiction.

Step 4: for all $\xi \in A^n$, $d_1(\xi) \in \{\xi_1, \dots, \xi_n\}$. Suppose not: for some $x \in A$ and $\xi \in A^n$, $d_1(\xi) = x \notin \{\xi_1, \dots, \xi_n\}$. Choose $P \in L^n$ such that ${}^1P = \xi$ and ${}^mP = (x^N)$. By DOWN_1 , ${}^1f(P) = d_1({}^1P) = d_1(\xi) = x$. By UP_1 and step 1, ${}^mf(P) = u_1({}^mP) = u_1(x^N) = x$: contradiction.

Step 5: there are $i \in N$, $x \in A$ and $y \in A \setminus \{x\}$ such that $d_1(x^i, y^{-i}) = x$. Suppose not:

$$\text{for all } i \in N, x \in A \text{ and } y \in A \setminus \{x\}, d_1(x^i, y^{-i}) = y. \quad (1)$$

Choose $r \in \{2, \dots, n-1\}$ and, arguing inductively, assume that, for all $x \in A$, $y \in A \setminus \{x\}$ and $I \subset N$ with r members, $d_1(x^I, y^{-I}) = y$. To eventually reach a contradiction of (1), it suffices to show that for all $x \in A$, $y \in A \setminus \{x\}$ and $I \subseteq N$ with $r+1$ members, $d_1(x^I, y^{-I}) = y$. So choose $x \in A$, $y \in A \setminus \{x\}$, $I \subset N$ with r members and $i \in N \setminus I$. Let $P \in L^n$ satisfy ${}^1P = (x^I, y^{-I})$ and ${}^mP = (y^I, x^i, z^{-(I \cup \{i\})})$, where $z \in A \setminus \{x, y\}$. By DOWN_1 , ${}^1f(P) = d_1({}^1P) = d_1(x^I, y^{-I})$. By the induction hypothesis, $d_1(x^I, y^{-I}) = y$. By UP_1 and step 3, ${}^mf(P) = u_1({}^mP) \in \{x, y, z\}$. Given that ${}^1f(P) = y$, it follows that ${}^mf(P) \in \{x, z\}$.

Case 1: ${}^mf(P) = x$. Consider any $Q \in L^n$ such that: (i) ${}^1Q = (z^i, x^{-i})$; and (ii) ${}^mQ = {}^mP$. By UP_1 , ${}^mf(Q) = u_1({}^mQ) = u_1({}^mP) = {}^mf(P) = x$. By DOWN_1 and step 4, ${}^1f(Q) = d_1({}^1Q) = d_1(z^i, x^{-i}) \in \{x, z\}$. Since ${}^mf(Q) = x$, ${}^1f(Q) = z$. Hence, $d_1(z^i, x^{-i}) = z$, contradicting (1).

Case 2: ${}^mf(P) = z$. With $J = I \cup \{i\}$, let $R \in L^n$ satisfy: (i) ${}^1R = (z^J, y^{-J})$; and (ii) ${}^mR = {}^mP$. By UP_1 , ${}^mf(R) = u_1({}^mR) = u_1({}^mP) = {}^mf(P) = z$. By DOWN_1 and step 4, ${}^1f(R) = d_1({}^1R) = d_1(z^J, y^{-J}) \in \{y, z\}$. Since ${}^mf(R) = z$, ${}^1f(R) = y$. Choose next $S \in L^n$ such that: (i) ${}^1S = {}^1R$; and (ii) ${}^mS = (y^J, x^{-J})$. By DOWN_1 , ${}^1f(S) = d_1({}^1S) = d_1({}^1R) = {}^1f(R) = y$. By UP_1 and step 3, ${}^mf(S) = u_1({}^mS) = u_1(y^J, x^{-J}) \in \{x, y\}$. As ${}^1f(S) = y$, it follows that ${}^mf(S) = x$. Finally, choose $T \in L^n$ with ${}^1T = (x^J, y^{-J})$ and ${}^mT = {}^mS$. By UP_1 , ${}^mf(T) = u_1({}^mT) = u_1({}^mS) = {}^mf(S) = x$. By DOWN_1 and step 4, ${}^1f(T) = d_1({}^1T) = d_1(x^J, y^{-J}) \in \{x, y\}$. It then follows from ${}^mf(T) = x$ that ${}^1f(T) = y$. Thus, $d_1(x^J, y^{-J}) = y$, where J has $r+1$ members. ■

Proof of Proposition 4.1. “ $\Leftarrow$ ” Let f be dictatorial, so there is $i \in N$ such that, for all $P \in L^n$, $f(P) = P_i$. It is then plain that, for all $r \in \{1, \dots, m\}$, UNA, $\text{IND}_r^\downarrow$ and $\text{IND}_r^\uparrow$ hold. “ $\Rightarrow$ ” Since $r \geq 1$, $\text{IND}_1^\downarrow$ holds and, therefore, DOWN_1 also holds. Since $m - r - 1 \geq m -$

$(m-2)-1=1$, $\text{IND}_1^\uparrow$ holds and, therefore, UP_1 also holds. By Lemma 5.1, there are $i \in N$, $x \in A$ and $y \in A \setminus \{x\}$ such that $d_1(x^i, y^{-i}) = x$. Let $Q \in L^n$ satisfy ${}^1Q = (x^i, y^{-i})$.

• Step 1: for all $P \in L^n$, ${}^1f(P) = {}^1P_i$. Choose $P \in L^n$. Case 1: ${}^1P_i = v \neq x$. To show that ${}^1f(P) = v$, suppose otherwise: for some $z \in A \setminus \{v\}$, ${}^1f(P) = z$. Define $I = \{k \in N: {}^1P_k = x\}$ and $J = \{k \in N: {}^1P_k = v\}$. Case 1a: $I = \emptyset$. By DOWN_1 and Lemma 5.1(i), $z \neq x$. Choose $R \in L^n$ such that: (i) ${}^1R = {}^1Q$; (ii) ${}^mR = (z^i, x^{-i})$. By DOWN_1 , ${}^1f(R) = d_1({}^1R) = d_1({}^1Q) = d_1(x^i, y^{-i}) = x$. By UP_1 and Lemma 5.1(i), ${}^mf(R) = u_1({}^mR) = u_1(z^i, x^{-i}) \in \{x, z\}$. Since ${}^1f(R) = x$, ${}^mf(R) = z$. As a result, $u_1(z^i, x^{-i}) = z$. Let $S \in L^n$ satisfy ${}^1S = {}^1P$ and ${}^mS = {}^mR$. By UP_1 , ${}^mf(S) = u_1({}^mS) = u_1(z^i, x^{-i}) = z$. By DOWN_1 , ${}^1f(S) = d_1({}^1S) = d_1({}^1P) = {}^1f(P) = z$: contradiction. Case 1b: $I \neq \emptyset$. As ${}^1P_i \neq x$, $I \neq N$. Choose $T \in L^n$ with ${}^1T = (v^J, z^{-J})$ and ${}^mT = (z^J, v^{-J})$. By case 1a, for all $Z \in L^n$ such that ${}^1Z_i \neq x$ and $\{k \in N: {}^1Z_k = x\} = \emptyset$, ${}^1f(Z) = {}^1Z_i$. In particular, ${}^1f(T) = {}^1T_i = v$. By UP_1 and Lemma 5.1(i), ${}^mf(T) = u_1({}^mT) = u_1(z^J, v^{-J}) \in \{v, z\}$. Given ${}^1f(T) = v$, it must be that ${}^mf(T) = z$. Let $V \in L^n$ satisfy ${}^1V = {}^1P$ and ${}^mV = {}^mT$. By UP_1 , ${}^mf(V) = u_1({}^mV) = u_1({}^mT) = {}^mf(T) = z$. By DOWN_1 , ${}^1f(V) = d_1({}^1V) = d_1({}^1P) = {}^1f(P) = z$: contradiction.

Case 2: ${}^1P_i = x$. To show that ${}^1f(P) = x$, suppose otherwise: for some $z \in A \setminus \{x\}$, ${}^1f(P) = z$. Choose $w \in A \setminus \{x, z\}$ and define $K = \{k \in N: {}^1P_k = w\} \cup \{i\}$. Let $W \in L^n$ satisfy ${}^1W = {}^1P$ and ${}^mW = (z^K, w^{-K})$. By DOWN_1 , ${}^1f(W) = d_1({}^1W) = d_1({}^1P) = {}^1f(P) = z$. By UP_1 and Lemma 5.1(i), ${}^mf(W) = u_1({}^mW) = u_1(z^K, w^{-K}) \in \{w, z\}$. As ${}^1f(W) = z$, ${}^mf(W) = w$. Choose $X \in L^n$ such that ${}^1X = (w^K, z^{-K})$ and ${}^mX = {}^mW$. By UP_1 , ${}^mf(X) = u_1({}^mX) = u_1({}^mW) = {}^mf(W) = w$. By case 1, for all $Z \in L^n$ such that ${}^1Z_i \neq x$, ${}^1f(Z) = {}^1Z_i$. In particular, ${}^1f(X) = {}^1X_i = w$: contradiction.

• Step 2: for all $P \in L^n$, ${}^mf(P) = {}^mP_i$. Choose $P \in L^n$ and suppose ${}^mf(P) \neq {}^mP_i$. Let $Q \in L^n$ satisfy ${}^mQ = {}^mP$ and ${}^1Q_i = {}^mf(P)$. By UP_1 , ${}^mf(Q) = u_1({}^mQ) = u_1({}^mP) = {}^mf(P)$. By step 1, ${}^1f(Q) = {}^1Q_i = {}^mf(P)$: contradiction.

• Step 3: for all $P \in L^n$ and $t \in \{1, \dots, r\}$, ${}^tf(P) = {}^tP_i$. Step 1 proves the case $r = 1$. For $r \geq 2$, select $t \in \{2, \dots, r\}$ and, arguing inductively, suppose that, for all $P \in L^n$ and $s \in \{1, \dots, t-1\}$, ${}^sf(P) = {}^sP_i$. Choose $P \in L^n$. To prove that ${}^tf(P) = {}^tP_i$, suppose ${}^tf(P) \neq {}^tP_i$. With ${}^tf(P) = y$, the induction hypothesis implies $y \notin \{{}^1P_i, \dots, {}^{t-1}P_i\}$. Hence, as $m-r \geq 2$, there is $u > t$ such that ${}^uP_i = y$. Let $Q \in L^n$ be obtained from P by moving y in P_i from position u to position m , if necessary. Thus, for all $k \in N$ and $v \in \{1, \dots, t\}$, ${}^vQ_k = {}^vP_k$. By $\text{IND}_r^\downarrow$, $y \in \{{}^1f(Q), \dots, {}^tf(Q)\}$. By step 2, ${}^mQ_i = y$ implies ${}^mf(Q) = y$: contradiction.

• Step 4: for all $P \in L^n$ and $t \in \{m, m-1, \dots, r+1\}$, ${}^t f(P) = {}^t P_i$. By step 3, it actually suffices to show that, for all $P \in L^n$ and $t \in \{m, m-1, \dots, r+2\}$, ${}^t f(P) = {}^t P_i$. If $m-r-1 = 1$ then $r = m-2$ and step 2 proves the result. If $m-r-1 \geq 2$ then select $t \in \{m-1, \dots, r+2\}$ and, arguing inductively, suppose that, for all $P \in L^n$ and $s \in \{m, \dots, t+1\}$, ${}^s f(P) = {}^s P_i$. Choose $P \in L^n$. To prove that ${}^t f(P) = {}^t P_i$, suppose ${}^t f(P) \neq {}^t P_i$. With ${}^t f(P) = y$, the induction hypothesis implies $y \notin \{{}^{t+1} P_i, \dots, {}^m P_i\}$. Hence, as $t \leq r+2$, there is $u < t$ such that ${}^u P_i = y$. Let $Q \in L^n$ be obtained from P by moving y in P_i from position u to position 1. Thus, for all $k \in N$ and $v \in \{m, \dots, t\}$, ${}^v Q_k = {}^v P_k$. By $\text{IND}_{m-r-1}^\downarrow$, $y \in \{{}^t f(Q), \dots, {}^m f(Q)\}$. By step 1, ${}^1 Q_i = y$ implies ${}^1 f(Q) = y$: contradiction. ■

Lemma 5.2. With $n \geq 2 < m$, if $f: L^n \rightarrow L$ satisfies ONTO_1 , UP_1 and DOWN_{m-1} then, for all $P \in L^n$, ${}^1 f(P) \in \{{}^1 P_1, \dots, {}^1 P_n\}$.

Proof. Let u_1 be the mapping associated with UP_1 and $d_1, \dots, d_{m-1}$ those associated with DOWN_{m-1} . Step 1: for all $x \in A$, $u_1(x^N) = x$. Suppose not: for some $x \in A$ and $y \in A \setminus \{x\}$, $u_1(x^N) = y$. Let $P \in L^n$ be such that ${}^1 P = (y^N)$ and ${}^m P = (x^N)$. By UP_1 , ${}^m f(P) = u_1(x^N) = y$. Therefore, by DOWN_{m-1} , for all $r \in \{1, \dots, m-1\}$, ${}^r f(P) = d_r(y^N) \neq y$. So to contradict ONTO_1 , it suffices, by UP_1 , to show that, for all $\xi \in A^n \setminus \{(y^N)\}$, $d_1(\xi) \neq y$. Choose $\xi \neq (y^N)$ and $z \in A \setminus \{x, y\}$. With $I = \{i \in N: \xi_i = x\}$, let $Q \in L^n$ satisfy ${}^1 Q = {}^1 P$, ${}^m Q = (z^I, x^{-I})$ if $I \neq \emptyset$ and ${}^m Q = (x^N)$ if $I = \emptyset$. By DOWN_1 , ${}^1 f(Q) = d_1(x^N) = {}^1 f(P) \neq y$. Hence, by DOWN_{m-1} , for all $r \in \{2, \dots, m-1\}$, ${}^r f(Q) = d_r(y^N) = {}^r f(P) \neq y$. Accordingly, ${}^m f(Q) = y$. By UP_1 , $u_1({}^m Q) = y$. Let $R \in L^n$ satisfy ${}^1 R = \xi$ and ${}^m R = {}^m Q$. By UP_1 , ${}^m f(R) = u_1({}^m R) = u_1({}^m Q) = y$. As a result, ${}^1 f(R) \neq y$. By DOWN_1 , $y \neq {}^1 f(R) = d_1({}^1 R) = d_1(\xi)$. Step 2: for all $P \in L^n$, ${}^1 f(P) \in \{{}^1 P_1, \dots, {}^1 P_n\}$. Suppose not: $x \in A$ and $P \in L^n$ satisfy ${}^1 f(P) = x \notin \{{}^1 P_1, \dots, {}^1 P_n\}$. Choose $Q \in L^n$ such that ${}^1 Q = {}^1 P$ and ${}^m Q = (x, \dots, x)$. By DOWN_1 , ${}^1 f(Q) = d_1({}^1 Q) = d_1({}^1 P) = {}^1 f(P) = x$. But, by UP_1 and step 1, ${}^m f(Q) = u_1({}^m Q) = x$: contradiction. ■

Proof of Proposition 4.7. “ $\Leftarrow$ ” Let f be dictatorial, so there is $i \in N$ such that, for all $P \in L^n$, $f(P) = P_i$. It is evident that ONTO_1 holds. UP_1 holds with u_1 such that, for all $\xi \in A^n$, $u_1(\xi) = \xi_i$. And DOWN_{m-1} holds with mappings $d_1, \dots, d_{m-1}$ such that, for all $r \in \{1, \dots, m-1\}$ and $\xi \in A^n$, $d_r(\xi) = \xi_i$. “ $\Rightarrow$ ” By Lemma 5.2, UNA holds. DOWN_{m-1} implies DOWN_{m-2} , which implies $\text{IND}_{m-2}^\downarrow$. UP_1 is equivalent to $\text{IND}_1^\uparrow$. Hence, the result follows from Proposition 4.1. ■

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