A simple sequential mechanism to allocate objects

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Abstract
The paper deals with the implementation of allocation rules in Nash equilibria when goods are indivisible. A mechanism that allocates goods sequentially is suggested. The individuals' messages consist of stating whether they want to receive a good or not. Goods are assigned in some given order. The outcome function assigns the good under consideration to some of the individuals asking for it and that have not received yet a good. The only allocation rules that are implementable by this mechanism in Nash equilibria are those having a hierarchy of dictators. The same result is obtained if weak implementability replaces implementability and the mechanism generates efficient sets of Nash equilibria and preserve Nash equilibria under changes of the order in which the objects are assigned.

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1. Introduction

This paper considers the problem of implementing allocation rules for heterogeneous indivisible goods; see Jackson (2001) for an introduction to the theory of implementation and Maniquet (2003), and references therein, for implementation in economic environments in which goods are perfectly divisible.

Shapley and Scarf (1974) consider an allocation model, the Shapley-Scarf housing market, in which each individual owns a heterogeneous indivisible good (a house) and is willing to trade his own good for another good. See Kamijo and Kawasaki (2009, pp. 3-5) for a summary of basic results of this model and Pápai (2007) for a generalization of the model that allows individuals to be endowed with a set of goods.

The model adopted here to study allocation rules for indivisible goods is based on the Shapley-Scarf model. There is a finite set of n individuals and a finite set of $m \geq n$ indivisible heterogeneous goods (or objects) to be allocated among the individuals, who have strict preferences over the goods but do not individually own them. An allocation is an assignment of one good to each individual. An allocation rule is a mapping that outputs an allocation taking as input the individuals' preferences.

The most stringent result for allocation rules establishes conditions under which the rule can be understood as operating following a hierarchy of dictators (or serial dictatorship). This means that there is an individual who is always assigned his most preferred object, next another individual who is always assigned his most preferred object among those left by the first individual, and so on. Svensson (1999), Ergin (2000) and Pápai (2001) obtain this result, the latter for the case in which individuals may be assigned a set of goods. Ehlers (2002) and Ehlers and Klaus (2003) characterize rules having a hierarchy of diarchies, where each level of the hierarchy may have either one or two individuals instead of only one. Pápai (2000) characterizes a more general hierarchical structure having the hierarchy of diarchies as a particular case.

This paper shows that a hierarchy of dictators arises from the mere assumption that the allocation rule can be implemented by a certain sequential allocation mechanism via Nash (1951) equilibria. The mechanism is based on what appears to be a simple and natural way of allocating goods: proceed good by good, ask individuals whether they want it or not, and, finally, given what individuals have expressed, assign the good to one of the individuals that have asked for it, provided the selected individual has not previously received a good.

Implementability of an allocation rule f by a mechanism via Nash equilibria means that the outcome of each Nash equilibrium in the game $G(P)$ induced by the mechanism when the preference profile is P coincides with the allocation $f(P)$ generated by f with preferences P. Weak implementability just requires that the outcome of some Nash equilibrium of $G(P)$ agrees with $f(P)$. It is also shown that weak implementability leads to a hierarchy of dictators if the mechanism satisfies conditions of efficiency and symmetry. Efficiency asserts that the mechanism does not generate a Nash equilibrium that is Pareto dominated by another Nash equilibrium. Symmetry states that every Nash equilibrium generated by the mechanism remains a Nash equilibrium if the mechanism is altered by changing the order in which the objects are assigned.

2. Definitions

Two finite non-empty sets will be assumed given throughout: a set I whose n members designate individuals and a set X whose m members represent objects or anything that can be assigned to the individuals. The symbol $0 \notin X$ will represent the absence of objects. A linear order on a set S with k elements is a sequence $(s_1, \dots, s_k)$ of members of S such that $S = \{s_1, \dots, s_k\}$. For linear order $\sigma = (s_1, \dots, s_k)$ and $r \in \{1, \dots, k\}$, ${}^r\sigma$ designates the rth element in the sequence σ.

Definition 2.1. A preference p on X is a linear order on $X \cup \{0\}$ such that ${}^{m+1}p = 0$. The set of preferences on X is denoted by L and the set of preference profiles (the set of ways of assigning preferences to individuals) by L^n.

Preferences on X presume that having an object is preferred to having none. In addition, preferences on X are strict: no member of X is indifferent to any other member of X. Specifically, in preference $p = (x_1, \dots, x_m, 0)$ on X, x_s is preferred to x_t if and only if $s < t$. Hence, for preference p and $r \in \{1, \dots, m\}$, ${}^rp$ is the rth most preferred object in p.

Definition 2.2. An allocation over X is a mapping $\alpha : I \to X$ such that, for all $i \in I$ and $j \in I \setminus \{i\}$, $\alpha(i) \neq \alpha(j)$. The set of allocations over X is denoted by A.

An allocation is a way of assigning some object to every individual, with the proviso that no object is assigned to two different individuals.

Definition 2.3. An allocation rule is a mapping $f : L^n \to A$ associating an allocation over X with every profile of preferences on X.

For $P \in L^n$ and $i \in I$, P_i designates the preference corresponding to individual i in preference profile P and, for allocation rule f, $f_i(P)$ designates the member of X that is assigned to individual i in allocation $f(P)$.

Definition 2.4. An allocation rule f has a hierarchy of dictators if there is a linear order $(i_1, \dots, i_n)$ on the set I of individuals such that, for all $P \in L^n$: (i) $f_{i_1}(P) = {}^1P_{i_1}$; and (ii) for all $r \in \{2, \dots, n\}$, $f_{i_r}(P)$ is the first element in P_{i_r} that belongs to the set $\{x \in X$: there is no $t < r$ such that $f_{i_t}(P) = x\}$.

When allocation rule f has a hierarchy of dictators $(i_1, \dots, i_n)$, it can be interpreted that each allocation $f(P)$ is obtained by assigning to i_1 his most preferred object according to P_{i_1}, assigning to i_2 his most preferred object in $X \backslash \{{}^1P_{i_1}\}$ according to P_{i_2}, and so on.

The following sequential allocation mechanism will be formalized next. Objects are arranged in an arbitrary linear order σ. For each object x, each individual must tell whether he wants x or not. Starting with ${}^1\sigma$, a choice rule g determines, among those individuals asking for ${}^1\sigma$, which individual is assigned ${}^1\sigma$. Proceeding next with ${}^2\sigma$, the same choice rule g determines, among those individuals asking for ${}^1\sigma$ that have not previously received any object, which individual is assigned ${}^2\sigma$. And so on until every individual obtains an object or all the objects have been considered.

Definition 2.5. For $i \in I$, the message set M_i of individual i is the set of functions $s_i : X \to \{\text{YES}, \text{NO}\}$ and $M = \prod_{i \in I} M_i$ is the set of profiles of messages.

Message sets M_i formalize the way in which individuals may ask for objects: for $s_i \in M_i$ and $x \in X$, $s_i(x) = \text{YES}$ indicates that individual i asks for object x, whereas $s_i(x) = \text{NO}$ means that i does not ask for object x.

Definition 2.6. Let I^* be the set of all non-empty subsets of I. A choice rule is a mapping $g : X \times I^* \to I$ such that, for all $x \in X$ and $J \in I^*$, $g(x, J) \in J$.

For every object x, if some individual in J must receive x, then the choice rule g selects the member of J getting x.

Definition 2.7. Let g be a choice rule, σ a linear order on X, and $s = (s_1, \dots, s_n)$ a profile of messages. Define $A_g^{\sigma}({}^1\sigma, s) = \{i \in I$: $s_i({}^1\sigma) = \text{YES}\}$ and, for $r \in \{2, \dots, m\}$, define $A_g^{\sigma}({}^r\sigma, s) = \{i \in I$: $s_i({}^r\sigma) = \text{YES}$ and, for all $k \in \{1, \dots, r-1\}$, $g({}^k\sigma, A_g^{\sigma}({}^k\sigma, s)) \neq i\}$

The set $A_g^\sigma({}^r\sigma, s)$ is the set of active individuals when object ${}^r\sigma$ is been assigned under profile s of messages. In particular, $A_g^\sigma({}^1\sigma, s)$ is the set of individuals asking for the first object ${}^1\sigma$ to be allocated, whereas, for $r \geq 2$, $A_g^\sigma({}^r\sigma, s)$ is the set of individuals that have not received yet an object and ask for the rth object ${}^r\sigma$ to be allocated. For $r \geq 2$, $A_g^\sigma({}^r\sigma, s)$ cannot include individuals asking for ${}^r\sigma$ that have already received some object to prevent the possibility that the same individual gets more than one object.

Definition 2.8. A sequential allocation mechanism is a triple (I, M, C_g^σ) consisting of: (i) a set I of individuals; (ii) a set $M = \prod_{i\in I} M_i$ of profile messages; and (iii) an outcome function $C_g^\sigma : I \times M \to X \cup \{0\}$, defined from a choice rule $g : X \times I^* \to I$ and a linear order $\sigma = (x_1, \ldots, x_m)$ on X so that, for all $s \in M$ and $r \in \{1, \ldots, m\}$:

(a) if $A_g^\sigma({}^r\sigma, s) = \varnothing$, then, for all $i \in I$, $C_g^\sigma(i, s) \neq \sigma_r$;
(b) if $A_g^\sigma({}^r\sigma, s) \neq \varnothing$ and $g({}^r\sigma, A_g^\sigma({}^r\sigma, s)) = i$, then $C_g^\sigma(i, s) = {}^r\sigma$; and
(c) if, for all $x \in X$, $C_g^\sigma(i, s) \neq x$, then $C_g^\sigma(i, s) = 0$.

The function C_g^σ in the sequential allocation mechanism (I, M, C_g^σ) determines the outcome of the mechanism. This outcome depends on the order σ in which objects are allocated and the choice rule g selecting individuals. By (a), if no individual is active when object x is being allocated, then x is assigned to no individual. By (b), if some individual is active when object x is being allocated, then the object is assigned to the individual selected by the choice rule g. By (c), if the choice rule g does not assign an object to some individual then the individual gets no object.

Definition 2.9 next adds preferences on outcomes to the sequential allocation mechanism to transform it into a strategic (or normal form) game. It is worth noticing that, though the mechanism is sequential (objects are not allocated simultaneously but sequentially), the associated game is simultaneous. This is because the individuals' claims for objects (their strategies) are furnished before the allocation of goods starts.

Definition 2.9. For $P \in L^n$, the game associated with sequential allocation mechanism (I, M, C_g^σ) and preference profile P is the four-tuple $G = (I, M, C_g^\sigma, P)$ such that:

(i) I is the set of players of the game;
(ii) $M = \prod_{i\in I} M_i$ is the set of strategy profiles of the game, where, for each $i \in I$, the message set M_i is the set of strategies of player i;

(iii) $C_g^\sigma : I \times M \to X \cup \{0\}$ is the outcome function of the game such that, for all $i \in I$ and $s \in M$, $C_g^\sigma(i, s)$ is the outcome for player i when strategy profile s is played; and

(iv) P is a preference profile such that, for each $i \in I$, P_i is player i's preference on the set $X \cup \{0\}$ of outcomes of the game.

Given a sequential allocation mechanism (I, M, C_g^σ), let $C_g^\sigma(s)$ denote the distribution of objects among the individuals. When, for all $i \in I$, $C_g^\sigma(i, s) \neq 0$, $C_g^\sigma(s)$ is an allocation over X: the allocation α such that, for all $i \in I$, $\alpha(i) = C_g^\sigma(i, s)$. For strategy $t_i \in M_i$ of player i and strategy profile $s = (s_1, \ldots, s_n) \in M$, (t_i, s_{-i}) is the strategy profile obtained from s by replacing s_i with t_i.

Definition 2.10. With $G = (I, M, C_g^\sigma, P)$ being the game associated with the sequential allocation mechanism (I, M, C_g^σ) and the preference profile $P \in L^n$, strategy profile $s \in M$ is a Nash equilibrium of G if, for all $i \in I$ and $t_i \in M_i$, it is not the case that $C_g^\sigma(i, (t_i, s_{-i}))\ P_i\ C_g^\sigma(i, s)$.

Definition 2.11. With $G = (I, M, C_g^\sigma, P)$ being the game associated with the sequential allocation mechanism (I, M, C_g^σ) and the preference profile $P \in L^n$, a non-empty set $S \subseteq M$ of strategy profiles is efficient if, for each $s \in S$, there is no $s' \in S$ such that: (i) for some $i \in I$, $C_g^\sigma(i, s')\ P_i\ C_g^\sigma(i, s)$; and (ii) for all $i \in I$, it is not the case that $C_g^\sigma(i, s)\ P_i\ C_g^\sigma(i, s')$.

A set of strategy profiles S is efficient if no outcome is Pareto inefficient within the set of outcomes associated with S. In other words, the outcome of no member of S is Pareto dominated by the outcome of another member of S.

Definition 2.12. A sequential allocation mechanism (I, M, C_g^σ) implements allocation rule f in Nash equilibria if: (i) the game $G = (I, M, C_g^\sigma, P)$ has some Nash equilibrium; and (ii) for each $P \in L^n$ and each Nash equilibrium s of G, $f(P) = C_g^\sigma(s)$.

Definition 2.13. A sequential allocation mechanism (I, M, C_g^σ) weakly implements allocation rule f in Nash equilibria if, for every $P \in L^n$, there is some Nash equilibrium s of the game (I, M, C_g^σ, P) such that $f(P) = C_g^\sigma(s)$.

Implementation of an allocation rule f requires that, for every preference profile P, the outcome of each Nash equilibrium of the game associated with the sequential allocation mechanism and P coincides with the allocation $f(P)$ generated by f. Weak implementation requires this to be the case for the outcome of some Nash equilibrium.

Definition 2.14. A sequential allocation mechanism (I, M, C_g^σ) is efficient if, for each $P \in L^n$, the set of Nash equilibria of the game (I, M, C_g^σ, P) is efficient.

A sequential allocation mechanism is efficient if, in every game induced by the mechanism, the outcome of no Nash equilibrium is Pareto dominated by the outcome of another Nash equilibrium. This does not exclude the possibility that the outcome of a Nash equilibrium be Pareto dominated by the outcome of a strategy profile that does not constitute a Nash equilibrium.

Definition 2.15. A sequential allocation mechanism (I, M, C_g^σ) is symmetric if, for each $P \in L^n$, linear order τ on X, and Nash equilibrium s of the game (I, M, C_g^σ, P), s is also a Nash equilibrium of the game (I, M, C_g^τ, P).

In any game induced by a symmetric allocation mechanism, the Nash equilibria of the game are preserved under changes in the order in which objects are assigned.

3. Results

Proposition 3.1. For $m \geq n$, there is a sequential allocation mechanism that implements allocation rule f in Nash equilibria if and only if f has a hierarchy of dictators.

Remark 3.2. That a sequential allocation mechanism weakly implements allocation rule f in Nash equilibria is not sufficient for f to have a hierarchy of dictators when $m \geq n$.

With $I = \{1, 2, 3\}$ and $X = \{x, y, z\}$, define the allocation rule f to be such that: (i) for $P \in L^n$ such that $P_2 = P_3$, $z\ P_1\ x\ P_1\ y$, and $x\ P_2\ y\ P_2\ z$, $f_1(P) = z$, $f_2(P) = y$ and $f_3(P) = x$; and (ii) otherwise, the hierarchy (1, 2, 3) determines the allocation. Although f does not have a hierarchy of dictators, the following sequential allocation mechanism weakly implements f. With linear order $\sigma = (x, y, z)$ on X, let g be the choice rule such that, for all $v \in X$, $g(v, I) = g(v, \{3\}) = 3$, $g(v, \{2, 3\}) = g(v, \{2\}) = 2$, and $g(v, \{1, 2\}) = g(v, \{1, 3\}) = g(v, \{1\}) = 1$. For $Q \in L^n \setminus \{P\}$, the following profile $s \in M$ is a Nash equilibrium of the game (I, M, C_g^σ, Q) such that $f(Q) = C_g^\sigma(s)$: for all $i \in I$ and $v \in X$, $s_i(v) =$ YES if and only if $v = f_i(Q)$. And $s_1(x) = s_2(x) = s_3(x) = s_2(y) = s_1(z) =$ YES and $s_1(y) = s_3(y) = s_2(z) = s_3(z) =$ NO defines a Nash equilibrium s of (I, M, C_g^σ, P) such that $f(P) = C_g^\sigma(s)$.

Remark 3.3. That a sequential allocation mechanism implements allocation rule f in Nash equilibria is not sufficient for f to have a hierarchy of dictators when $m < n$.

With $I = \{1, 2, 3\}$ and $X = \{x, y\}$, define the allocation rule f to be such that, for all $P \in L^n$: (i) if ${}^1P_1 = x$, then $f_1(P) = x$ and $f_2(P) = y$; and (ii) if ${}^1P_1 = y$, then $f_1(P) = y$ and $f_3(P) = x$. Let (I, M, C_g^{σ}) the sequential allocation mechanism such that $\sigma = (x, y)$, $g(x, \{2, 3\}) = 3$, $g(y, \{2, 3\}) = 2$ and, for all $v \in X$, $g(v, I) = g(v, \{1\}) = g(v, \{1, 2\}) = g(v, \{1, 3\}) = 1$, $g(v, \{2\}) = 2$, and $g(v, \{3\}) = 3$. Whereas f does not have a hierarchy of dictators, (I, M, C_g^{σ}) does implement f in Nash equilibria. In fact, given $P \in L^n$, the following profile $s \in M$ is a Nash equilibrium of the game (I, M, C_g^{σ}, P) such that $f(P) = C_g^{\sigma}(s)$: for all $v \in X$, $s_3(v) = s_2(v) =$ YES and $s_1(v) =$ YES if and only if $v = f_1(P)$.

Proposition 3.4. For $m \geq n$, there is some efficient and symmetric sequential allocation mechanism (I, M, C_g^{σ}) that weakly implements allocation rule f in Nash equilibria if and only if f has a hierarchy of dictators.

4. Proofs

Given a choice rule g and $x \in X$, define $\to^x$ to be the binary relation on I such that, for all $i \in I$ and $j \in I$, $i \to^x j$ if and only if $g(x, \{i, j\}) = i$ and $i \neq j$.

Lemma 4.1. With $m \geq n$ and $\sigma = (x_1, \ldots, x_{m-1}, x)$, if (I, M, C_g^{σ}) is a sequential allocation mechanism that

(a) implements allocation rule f in Nash equilibria or
(b) is efficient and weakly implements allocation rule f in Nash equilibria,

then $\to^x$ is a linear order on I.

Proof. Let $\sigma = (x_1, \ldots, x_{m-1}, x)$. By definition, for all $i \in I$, it is not the case that $i \to^x i$. By definition of g, for all $i \in I$ and $j \in I\backslash\{i\}$, either $i \to^x j$ or $j \to^x i$. It then remains to be shown that there is no cycle $i \to^x j \to^x k \to^x i$, where $i \in I$, $j \in I\backslash\{i\}$ and $k \in I\backslash\{i, j\}$. To this end, suppose $i \to^x j \to^x k \to^x i$. The aim is to reach a contradiction. With $(i_1, \ldots, i_{n-3})$ being any linear order on $I\backslash\{i, j, k\}$, let $P \in L^n$ be any preference profile such that: (i) for all $i_r \in I\backslash\{i, j, k\}$, ${}^1P_{i_r} = x_r$ and ${}^mP_{i_r} = x$; and (ii) for all $h \in \{i, j, k\}$, ${}^1P_h = x$, ${}^2P_h = y$ and ${}^3P_h = z$. Let $G = (I, M, C_g^{\sigma}, P)$ be the game associated with (I, M, C_g^{σ}) and P.

Case 1: for all $h \in I$, $f_h(P) \neq x$. If $m = n$ then there must be $h \in I$ such that $f_h(P) = \varnothing$, contradicting the fact that $f(P)$ is an allocation over X. If $m > n$ then, as $f(P)$ is an allocation over X, there is some $w \in X \backslash \{x\}$ such that $f_i(P) = w$. Consider any $s \in M$ with $f(P) = C_g^\sigma(s)$ and let s_i' be the strategy such that, for all $v \in X$, $s_i'(v) =$ YES if and only if $v = x$. Since, for all $h \in I$, $f_h(P) \neq x$, it follows that $A_g^\sigma(x, s) = \varnothing$. Accordingly, $C_g^\sigma(i, (s_i', s_{-i})) = x$. Given that $x \, P_i \, w$, s is not a Nash equilibrium of G. As a result, there is no Nash equilibrium s of G such that $f(P) = C_g^\sigma(s)$, contradicting both (a) and (b).

Case 2: for some $h \in I \backslash \{i, j, k\}$, $f_h(P) = x$. By implementability or weak implementability, let s be a Nash equilibrium of G such that $f(P) = C_g^\sigma(s)$. Define $Q \in L^n$ to be any preference profile such that: (i) for all $r \in I \backslash \{i, h\}$, ${}^1Q_r = f_r(P)$; (ii) ${}^1Q_h = {}^2Q_i = f_i(P)$; and (iii) ${}^1Q_i = {}^2Q_h = x$. Since ${}^1P_i = {}^mP_h = x$, s is also a Nash equilibrium of the game $G' = (I, M, C_g^\sigma, Q)$. Let s' be the strategy profile such that, for all $r \in I$ and $v \in X$, $s_r'(v) =$ YES if and only if $v = {}^1Q_r$. Each player of G' receives in $C_g^\sigma(s')$ his most preferred outcome. This makes s' a Nash equilibrium of G'. But $C_g^\sigma(h, s) = x \neq f_i(P) = C_g^\sigma(h, s')$, so s and s' are two Nash equilibria of G' that yield different outcomes, contradicting (a). On the other hand, for all $r \in I \backslash \{i, h\}$, r receives the same object in both $C_g^\sigma(s)$ and $C_g^\sigma(s')$, whereas, for $r \in \{i, h\}$, r prefers, according to Q_r, the object he receives in $C_g^\sigma(s')$ to the one received in $C_g^\sigma(s)$. Consequently, the set of Nash equilibria of (I, M, C_g^σ, Q) is not an efficient set, contradicting (b).

Case 3: for some $h \in \{i, j, k\}$, $f_h(P) = x$. By implementability or weak implementability, let s be a Nash equilibrium of G such that $f(P) = C_g^\sigma(s)$. Letting $J = A_g^\sigma(x, s)$, $f_h(P) = x$ implies $h \in J$. Case 3a: $J \backslash \{h\} \neq \varnothing$. Let $r \in J \backslash \{h\}$. Then, being x the last object assigned, for all $v \in X \backslash \{x\}$, $f_r(P) \neq v$. Hence, $f_h(P) = x$ implies $f_r(P) \neq x$. Consequently, $f_r(P) = \varnothing$, which contradicts the fact that $f(P)$ is an allocation over X. Case 3b: $J = \{h\}$. It then follows from $i \to^x j \to^x k \to^x i$ that, for some $r \in \{i, j, k\}$, $r \to^x h$. Let strategy t_r satisfy, for all $v \in X$, $t_r(v) =$ YES if and only if $v = x$. Therefore, $C_g^\sigma(r, (t_r, s_{-r})) = x \neq C_g^\sigma(r, s)$ and, as a result, s is not a Nash equilibrium of G: contradiction.■

Lemma 4.2. With $m \geq n$, if (I, M, C_g^σ) is a sequential allocation mechanism that

(a) implements allocation rule f in Nash equilibria or
(b) is efficient and weakly implements allocation rule f in Nash equilibria,

then there is a linear order τ on I such that, for all $x \in X$, $\to^x$ is equal to τ.

Proof. With x being the object ranked last in σ, by Lemma 4.1, $\rightarrow^x$ is a linear order. Let $y \in X\backslash\{x\}$. The proof amounts to showing that $\rightarrow^y$ is equal to $\rightarrow^x$. Choose $i \in I$ and $j \in I\backslash\{i\}$. Suppose $i \rightarrow^x j \rightarrow^y i$. With $I\backslash\{i, j\} = \{i_1, \ldots, i_{n-2}\}$ and $X\backslash\{x, y\} = \{x_3, \ldots, x_m\}$, let $P \in L^n$ be any preference profile such that: (i) for all $i_r \in I\backslash\{i, j\}$, ${}^1P_{i_r} = x_r$; (ii) ${}^1P_i = {}^2P_j = y$; and (iii) ${}^1P_j = {}^2P_i = x$. Let $G = (I, M, C_g^\sigma, P)$ be the game associated with (I, M, C_g^σ) and P. Define s to be the strategy profile such that, for all $k \in I$ and $v \in X$, $s_k(v) =$ YES if and only if $v = {}^1P_k$. Since every player obtains in $C_g^\sigma(s)$ his most preferred object, s is a Nash equilibrium of G.

Let s' be the strategy profile such that: (i) for all $k \in I\backslash\{i, j\}$, $s_k' = s_k$; (ii) for all $k \in \{i, j\}$ and $v \in X$, $s_k'(v) =$ YES if and only if $v = {}^2P_k$. In this case, for all $k \in I\backslash\{i, j\}$, $C_g^\sigma(k, s') = {}^1P_k$, whereas, for all $k \in \{i, j\}$, $C_g^\sigma(k, s') = {}^2P_k$. Strategy profile s' is a Nash equilibrium of G because: (i) each player in $I\backslash\{i, j\}$ obtains in $C_g^\sigma(s')$ his most preferred object; (ii) player i obtains his second most preferred object x and, since $A_g^\sigma(y, s') = \{j\}$ and $g(y, \{i, j\}) = j$, player i cannot obtain his most preferred object y by changing his strategy s_i'; and (iii) player j obtains his second most preferred object y and, since $A_g^\sigma(x, s') = \{i\}$ and $g(x, \{i, j\}) = i$, player j cannot obtain his most preferred object x by changing his strategy s_j'. In sum, s and s' are two Nash equilibria of G that do not yield the same outcome, which contradicts (a). As regards (b), for all $k \in I\backslash\{i, j\}$, k receives the same object in both $C_g^\sigma(s)$ and $C_g^\sigma(s')$, whereas, for $k \in \{i, j\}$, k prefers, according to P_k, the object he receives in $C_g^\sigma(s)$ to the one obtained in $C_g^\sigma(s')$. As a consequence, the set of Nash equilibria of G is not an efficient set, contradicting (b).■

Proof of Propositions 3.1 and 3.4. "$\Rightarrow$" The proof is immediate if $n = 1$, so let $n \geq 2$. By Lemmas 4.1 and 4.2, for all $x \in X$, $\rightarrow^x$ is equal to some linear order $\rightarrow$ on I. It will be shown that $\rightarrow$ defines a hierarchy of dictators. Choose $P \in L^n$, $i \in I$ and $j \in I\backslash\{i\}$. Without loss of generality, assume $i \rightarrow j$. Since $m \geq n$ ensures that every individual receives some object in $f(P)$, the proof amounts to showing that $f_j(P)\ P_i\ f_i(P)$ cannot be. Suppose otherwise: with $x = f_j(P)$ and $y = f_i(P)$, let $x\ P_i\ y$. By implementability or weak implementability, there is a mechanism (I, M, C_g^σ) such that, for some Nash equilibrium s of $G = (I, M, C_g^\sigma, P)$, $f(P) = C_g^\sigma(s)$. With s being one such Nash equilibrium, let $Q \in L^n$ satisfy: (i) for all $k \in I\backslash\{i, j\}$, ${}^1Q_k = f_k(P)$; (ii) ${}^1Q_i = x$ and ${}^2Q_i = y$; and (iii) $x\ P_j\ y$ implies ${}^1Q_j = x$ and ${}^2Q_j = y$, whereas $y\ P_j\ x$ implies ${}^1Q_j = y$ and ${}^2Q_j = x$. The fact that s is a Nash equilibrium of G implies that s is also a Nash equilibrium of $G' = (I, M, C_g^\sigma, Q)$. Let s' be the strategy profile such that: (i) for all $k \in I\backslash\{i, j\}$ and $v \in X$, $s_k'(v) =$ YES if and only if $v = {}^1Q_k$; (ii) for all $v \in X$, $s_i'(v) =$ YES if and only if $v = x$; and (iii) for all $v \in X$, $s_j'(v) =$ YES if and only if $v = y$.

• End of the proof of Proposition 3.1. The outcome $C_g^\sigma(s')$ generated by s' is such that each $k \in I\backslash\{j\}$ receives his most preferred object according to Q_k. If ${}^1Q_j = y$, then j also receives his most preferred object; and if ${}^1Q_j = x$, then, as $A_g^\sigma(x, s') = \{i\}$ and $i \to j$, there is no strategy of player j that, given s_{-i}', allows j to obtain x. Therefore, s' and s are two Nash equilibria of G' with different outcomes. This contradicts the assumption that (I, M, C_g^σ) implements f in Nash equilibria.

• End of the proof of Proposition 3.4. Case 1: ${}^1Q_j = y$. Then the set of Nash equilibria of G' is not efficient: s' and s are two Nash equilibria of G' such that, for all $k \in I\backslash\{i, j\}$, k is assigned the same object in both $C_g^\sigma(s)$ and $C_g^\sigma(s')$, whereas, for $k \in \{i, j\}$, k prefers, according to Q_k, the object he receives in $C_g^\sigma(s')$ to the one assigned in $C_g^\sigma(s)$. Case 2: ${}^1Q_j = x$. Define τ to be the linear order on X such that the first $n - 2$ objects in τ are those in the set $Z = \{z \in X$: for some $k \in I\backslash\{i, j\}$, $z = {}^1Q_k\}$ and such that y occupies the first position on τ restricted to $X\backslash Z$. Hence, τ assigns first the most preferred objects of the members of the set $I\backslash\{i, j\}$ and next assigns y. Since s is a Nash equilibrium of G', by symmetry, s must be a Nash equilibrium of $G'' = (I, M, C_g^\tau, Q)$. By efficiency, for all $k \in I\backslash\{i, j\}$, $C_g^\tau(k, s) = {}^1Q_k$. Consequently, when object y is going to be assigned following the order τ, $A_g^\tau(y, s) \subseteq \{i, j\}$: only i and j can be active players. The fact that s is a Nash equilibrium of G such that $f(P) = C_g^\sigma(s)$ implies that $C_g^\sigma(i, s) = f_i(P) = y$. Accordingly, $s_i(y) =$ YES. As a result, $i \in A_g^\tau(y, s)$. Since $i \to j$, $C_g^\tau(i, s) = y$. Hence, $i \notin A_g^\tau(x, s)$. But $i \to j$ ensures that $C_g^\tau(i, (s_i', s_{-i})) = x$. This proves that s is not a Nash equilibrium of G'': contradiction.

"$\Leftarrow$" With f being an allocation rule having a hierarchy of dictators $(i_1, \dots, i_n)$, let (I, M, C_g^σ) be the sequential allocation mechanism such that: (i) σ is any linear order on X; and (ii) for all non-empty $J \subseteq I$ and $x \in X$, $g(x, J)$ is the member of J appearing first in the list $(i_1, \dots, i_n)$. To prove that (I, M, C_g^σ) implements f in Nash equilibria, choose $P \in L^n$ and let $G = (I, M, C_g^\sigma, P)$ be the game associated with (I, M, C_g^σ) and P. The strategy profile s such that, for all $i \in I$ and $x \in X$, $s_i(x) =$ YES if and only if $x = f_i(P)$ is a Nash equilibrium of G satisfying $C_g^\sigma(s) = f(P)$. This proves (i) in Definition 2.12. With respect to (ii), suppose t is another Nash equilibrium of G. By definition of g, for all $x \in X$ and J such that $i_1 \in J$, $g(x, J) = i_1$. In view of this, s_{i_1} guarantees that the mechanism assigns $f_{i_1}(P)$ to i_1, so it cannot be that $C_g^\sigma(i_1, t) \neq C_g^\sigma(i_1, s)$. Taking this as the base case of an induction argument, choose $r \in \{2, \dots, m\}$, and suppose that, for all $k \in \{1, \dots, r - 1\}$, $C_g^\sigma(i_k, t) = C_g^\sigma(i_k, s) = f_{i_k}(P)$. To show that $C_g^\sigma(i_r, t) = C_g^\sigma(i_r, s)$, observe that the definition of g implies that, for no strategy t_{i_r}' of player i_r, $C_g^\sigma(i_r, (t_{i_r}', t_{-i_r})) \in \{f(P_{i_1}), \dots, f(P_{i_{r-1}})\}$. Hence, given the strategies of the rest of players in t, player i_r can only obtain objects in $X\backslash\{f(P_{i_1}), \dots, f(P_{i_{r-1}})\}$. As $f_{i_r}(P)$ is i_r's most preferred object in that set, the strategy s_{i_r} ensures that the mechanism assigns $f(P_{i_r})$ to i_r. As a result, it cannot be that $C_g^\sigma(i_r, t) \neq C_g^\sigma(i_r, s)$. To sum up, (I, M, C_g^σ) implements f in Nash equilibria. A fortiori, (I, M, C_g^σ) weakly implements f in Nash equilibria. Since, for each $P \in L^n$, all the Nash of equilibria of the game $G = (I, M, C_g^\sigma, P)$ generate the same outcome, the set of Nash equilibria of G is efficient, so (I, M, C_g^σ) is efficient. And since σ was arbitrary, (I, M, C_g^σ) is symmetric.■

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