Allocating objects using menus

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Abstract
The problem of allocating heterogeneous indivisible goods is considered when allocations are assumed to be the result of allowing each individual to choose his most preferred good from a menu of goods previously assigned to the individual. Such menus could be viewed as the equivalent, in an allocation model without prices, of competitive budget sets. It is shown that the allocation of goods is determined by a hierarchy of diarchies if menus: (i) cannot be enlarged when fewer goods have to be allocated; and (ii) cannot become smaller when fewer individuals participate in the allocation. The addition of a strategy-proof condition transforms the hierarchy of diarchies into a hierarchy of dictators.

Keywords: Allocation rule, hierarchy of dictators, hierarchy of diarchies, indivisible good, strict preference.

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1. Introduction

The competitive market model currently represents the quintessential allocation model. The first and second theorems of welfare economics bear witness to basic virtues of a system of competitive markets as an allocation mechanism; see, for instance, Mas-Colell, Whinston and Green (1995). In a system of competitive markets in which the only agents are consumers, the allocation of goods among them can be seen as the outcome of a two-stage process. At the first stage, each consumer is assigned a budget set, which depends on the prices of the goods and the consumer's endowment, which is a proxy for the consumer's income. At the second stage, each consumer chooses from his budget set one of the most preferred elements. A competitive equilibrium arises when, given endowments, the prices of goods are such that the choices consumers independently make are feasible and mutually consistent. It is the mutual independence and consistency of the consumers' choices in a competitive equilibrium that justifies the interpretation of competitive markets as decentralized allocation mechanisms.

This paper adapts the competitive budget set approach to the problem of allocating heterogeneous indivisible goods when there is no medium of exchange. This problem is analyzed in a modified version of the allocation model by Shapley and Scarf (1974). Specifically, given universal sets N of individuals and X of objects, an allocation problem will be given by three items: a subset B of X consisting of the goods that have to be allocated; a subset I of N consisting of the individuals among whom the objects in B have to be distributed; and the preferences on B of the individuals in I.

An allocation rule associates with each allocation problem an assignment of the objects to the individuals. To model the idea that the allocation rule embodies some form of decentralized allocation mechanism, each individual will be given a menu (or choice set) from which he will be entitled to choose, and be assigned, his most preferred object. Menus are a function of the same elements that characterize allocation problems: preferences, relevant individuals and available objects. Menus can then be viewed as a possible way of defining competitive budget sets in an allocation model without prices.

Two conditions are imposed on menus. One asserts that menus cannot be enlarged if fewer objects have to be allocated. The other states that menus cannot become smaller if fewer individuals participate in the allocation problem. The main result of the paper shows that the allocation rules that can be obtained with the help of menus satisfying those two properties are rules having a hierarchy of diarchies.

An allocation rule has a hierarchy of dictators when there is a list h of the members of the set N of individuals such that, for every allocation problem, the restriction $(i_1, \dots, i_r)$ of h to the set of individuals taking part in the allocation problem is such that i_1 is assigned his most preferred good, i_2 is assigned his most preferred good among those left by i_1, and so on. Characterizations of this type of allocation rule have been obtained by Svensson (1999), Ergin (2000) and Pápai (2001), the latter for the case in which individuals may be assigned a set of goods.

A hierarchy of diarchies is like a hierarchy of dictators with the difference that the elements in the list h, instead of consisting of just a single individual, may also consist of a pair of individuals. Two individuals, i and j, coexisting at the same level of the hierarchy still have priority over individuals at lower levels. The conflict that may arise when the available good that both prefer most is the same is resolved by letting i have priority over j for some goods and letting j have priority over i for the rest of goods. Ehlers (2002) and Ehlers and Klaus (2003) suggest axioms characterizing the rules having a hierarchy of diarchies; see Pápai (2000) for a characterization of hierarchical exchange rules, of which the hierarchy of diarchies is a particular case, and Ergin (2002) for a general analysis of allocation on the basis of priority structures.

The second result in the paper shows how to transform the hierarchy of diarchies into a hierarchy of dictators by imposing a condition on menus that can be interpreted as a non-manipulability requirement. This condition presumes that the preferences taken into account to define menus are provided by the individuals themselves and holds that an individual cannot enlarge his menu by reporting a false preference.

These results can be interpreted in the sense that decentralized allocation makes the allocation of goods be strongly dependent on the power of individuals. Or, perhaps more precisely, that the power of individuals can be exercised without much constraints when allocation is decentralized. This is evident when allocation rules have a hierarchy of dictators. But the competitive model itself allows the interpretation that the allocation is the result of individuals exercising their power, which in this case can be measured in terms of income. The hierarchical structure in the competitive model may be regarded as endogenous, because income depends on equilibrium prices. Hence, the hierarchy becomes a function of the preferences, something that does not occur in rules governed by hierarchies of diarchies or dictators. This notwithstanding, the allocation generated by competitive markets is still the result of some hierarchical structure determining the priority to obtain goods on the basis of income: goods are assigned to those willing to pay more, and willingness to pay is positively correlated with income.

2. Definitions and assumptions

The $n \geq 1$ members of the finite set N represent individuals. A society is a non-empty subset of N. The $m \geq 1$ members of the finite set X represent objects or anything that can be assigned to the individuals. The null object is represented by $0 \notin X$. A booty is a non-empty subset of X. A preference on booty B is a sequence $(x_1, \ldots , x_r)$ listing the r members of B without repetitions. The interpretation is that x_s is preferred to x_t if and only if $s < t$. The null object 0 is not included in the preference because it is presumed that every object is preferred to 0. For booty B, L_B is the set of preferences that can be defined on B. For $p \in L_B$ and booty $C \subseteq B$, the restriction $p|_C$ of p to C is the sequence obtained from p by removing all the members of $B\backslash C$. For preference p on B, ${}^1p$ designates the first element in the sequence p, that is, the most preferred object in B.

Definition 2.1. For society I and booty B, a preference profile based on I and B is a function $P_I^B : I \rightarrow L_B$. The set L is the set of all preference profiles $P_I^B : I \rightarrow L_B$, where I is a society and B is a booty.

A preference profile based on I and B is a function that assigns a preference on B to each member of the society I. For $P_I^B \in L$ and $i \in I$, P_i^B abbreviates $P_I^B(i)$. For $P_I^B \in L$, society $J \subseteq I$ and booty $C \subseteq B$, P_J^C designates the preference profile Q_J^C based on J and C such that, for all $i \in J$, $Q_i^C = P_i^B|_C$. In words, P_J^C is obtained from P_I^B by removing the individuals in $I\backslash J$ and restricting the preferences of each member of J to C.

Definition 2.2. For society I and booty B, an allocation based on I and B is a function $\alpha_I^B : I \rightarrow B \cup \{0\}$ such that: (i) for some $i \in I$, $\alpha_I^B(i) \in B$; and (ii) for all $i \in I$ and $j \in I$, if $\alpha_I^B(i) = \alpha_I^B(j) \in B$, then $i = j$. The set A is the set of all allocations $\alpha_I^B : I \rightarrow L_B$, where I is a society and B is a booty.

An allocation based on I and B is a way of assigning at least some member of B to some individual in I, with the proviso that the same object in B is not assigned to two different individuals in I.

Definition 2.3. An allocation rule is a function $f : L \rightarrow A$ such that, for all $P_I^B \in L$, $f(P_I^B)$ is an allocation based on I and B.

An allocation rule is a way of determining, for each society I and booty B, an allocation based on I and B from a preference profile based on I and B. Therefore, an allocation rule transforms preferences into allocations. For allocation $f(P_I^B)$ and $i \in I$, $f_i(P_I^B)$

designates the member of $B \cup \{0\}$ assigned to i: if $f_i(P_I^B) \in B$, then individual i is assigned object $f_i(P_I^B)$; and if $f_i(P_I^B) = 0$, then i is assigned no object.

Definition 2.4. A menu is a mapping $M : N \times L \to A$ such that, for all $P_I^B \in L$: (i) $i \notin I$ implies $M(i, P_I^B) = \varnothing$; (ii) $i \in I$ implies $M(i, P_I^B) \subseteq B$; and (iii) if B has at least two elements, then, for some $i \in I$, $M(i, P_I^B)$ has at least two elements.

Writing $M_i(P_I^B)$ instead of $M(i, P_I^B)$, the set $M_i(P_I^B)$ represents the set of objects from which i will be allowed to choose, in accordance with his preference P_i^B, when P_I^B defines the preferences on booty B of the individuals in society I. The term "menu" will refer both to the mapping M and to the choice sets $M_i(P_I^B)$.

Specifically, (i) holds that if i is not a member of society I, then, for all $P_I^B \in L$, $M_i(P_I^B) = \varnothing$, which means that i is not entitled to choose anything. In addition, (ii) requires that, for every member i of I, $M_i(P_I^B) \subseteq B$. This presumes that the only relevant preferences are those over objects that can be actually allocated, so when B represents the set of such objects, nobody should be entitled to choose anything not included in B. Finally, (iii) is based on the idea that menus should be granted to individuals with a minimum of generosity, since menus define a domain from which an individual is free to choose an object. If a menu is a singleton, then the menu does not properly define a choice set, but rather constitutes a dictated choice. Accordingly, to confer to at least one individual not only the freedom to choose but also the freedom to reject, some menu must contain at least two objects when the booty itself contains at least two objects.

Definition 2.5. A menu M can generate an allocation rule if, for all $P_I^B \in L$, $i \in I$ and $j \in I \backslash \{i\}$, ${}^1P_i|_{M_i(P_I^B)} \neq {}^1P_j|_{M_j(P_I^B)}$.

A menu M can generate an allocation rule when the menus assigned to the individuals are always consistent in the sense that two individuals would not choose the same object from their respective menus. This is one of the conditions defining a competitive equilibrium.

Definition 2.6. Let M be a menu that can generate an allocation rule. Then the allocation rule that M generates is the allocation rule f such that, for all $P_I^B \in L$ and $i \in I$, if $M_i(P_I^B) \neq \varnothing$, then $f_i(P_I^B) = {}^1P_i|_{M_i(P_I^B)}$.

When a menu M can generate some allocation rule, M can be viewed as a mechanism determining the outcome of a decentralized allocation process: individuals are awarded menus in such a way that, by letting each individual freely choose from his assigned menu, an allocation always result. A similar decentralization mechanism can be interpreted to be operating in exchange economies in which consumers are competitive and equilibrium allocations are the outcome of the mechanism.

Remark 2.7. Every allocation rule can be generated by several menus.

In fact, an allocation rule f is generated by every menu M such that, for all $P_I^B \in L$ and $i \in I$: (i) $f_i(P_I^B) = 0$ implies $M_i(P_I^B) = \varnothing$; and (ii) $f_i(P_I^B) \in B$ implies $M_i(P_I^B) = \{f_i(P_I^B)\} \cup C$, where $C \subseteq \{x \in B\colon f_i(P_I^B)$ is preferred to x in preference $P_i^B\}$.

Definition 2.8. For allocation rule f, society I and object $x \in A$, individual $i \in I$ dictates over (x, I) in f if, for all $P_J^B \in L$ such that $i \in J \subseteq I$ and ${}^1P_i^B = x$, $f_i(P_J^B) = x$.

Definition 2.9. Allocation rule f has a hierarchy of diarchies if there is a sequence $(I_1, \dots, I_r)$ of societies such that, for all $s \in \{1, \dots, r\}$ and $t \in \{1, \dots, r\}\backslash\{s\}$:

- (i) $I_1 \cup \dots \cup I_r = N$;
- (ii) $I_s \cap I_t = \varnothing$;
- (iii) I_s has either one or two members;
- (iv) if $I_s = \{i\}$, then, for all $x \in A$, i dictates over $(x, I_s \cup \dots \cup I_r)$ in f; and
- (v) if $I_s = \{i, j\}$, then, for some non-empty $B \subset A$, each $x \in B$ and each $y \in A\backslash B$,
  - (a) i dictates over $(x, I_s \cup \dots \cup I_r)$ and over $(y, I_s\backslash\{j\} \cup I_{s+1} \cup \dots \cup I_r)$ in f, and
  - (b) j dictates over $(y, I_s \cup \dots \cup I_r)$ and over $(x, I_s\backslash\{i\} \cup I_{s+1} \cup \dots \cup I_r)$ in f.

When an allocation rule has a hierarchy of diarchies, the allocation defined by the rule can be seen as the result of letting an exogenous hierarchical priority structure determine how objects are assigned. The sequence $(I_1, \dots, I_r)$ defines the priority structure when the allocation problem involves all the individuals and all the objects. Each level of the hierarchy has one or two individuals. Individuals at higher levels of the hierarchy have priority over the members at lower levels, so that a member of a lower level cannot be assigned an object a member of a higher level would prefer to obtain. And if i and j belong to the same level, i has priority over j for some objects and j has priority over i for the rest of objects. If the allocation problem involves society I and booty B, then the same procedure applies but considering instead the restriction of $(I_1, \dots, I_r)$ to I, which is the sequence obtained from $(I_1 \cap I, \dots, I_r \cap I)$ by removing all the occurrences of the empty set $\varnothing$.

Definition 2.10. Allocation rule f has a hierarchy of dictators if it has hierarchy of diarchies such that the corresponding sequence $(I_1, \dots, I_r)$ consists of singletons.

ROB. *How menus are affected by the removal of objects*.
For all $P_I^B \in L$, $i \in I$ and $x \in B$, $M_i(P_I^{B\setminus\{x\}}) \subseteq M_i(P_I^B)\setminus\{x\}$.

ROB is a property of non-expansiveness: if an object is removed from the booty, then menus cannot be enlarged. ROB requires that the collective misfortune caused by losing one of the objects of the booty cannot generate an individual benefit, represented by having the possibility of choosing from a larger menu.

RIN. *How menus are affected by the removal of individuals*.
For all $P_I^B \in L$, $i \in I$ and $j \in I\setminus\{i\}$, $M_i(P_I^B) \subseteq M_i(P_{I\setminus\{j\}}{}^B)$.

RIN is a property of non-contractiveness: if an individual is removed from a society, then the menus of the remaining individuals cannot shrink. RIN guarantees that the loss that for an individual represents being expelled from the allocation process generates a collective benefit that may be distributed among the remaining individuals in terms of larger menus.

SPR. *Strategy-proofness of the menu assignment process*.
For all $P_I^B \in L$, $i \in I$ and $Q_i^B \in L$, it is not the case that $M_i(P_I^B) \subset M_i(Q_i^B, P_{I\setminus\{i\}}{}^B)$, where $(Q_i^B, P_{I\setminus\{i\}}{}^B)$ is the member of L obtained from P^B by replacing P_i^B with Q_i^B.

SPR is based on the presumption that individuals reveal the preferences that determine the menu assignment. In this context, suppose that i's true preference is P_i^B. Then SPR states that i cannot enlarge his menu by reporting a false preference Q_i^B. Since it never harms to have a larger menu, SPR can be viewed as a condition of non-manipulability of the mechanism that assigns menus. In fact, to remove any incentive to manipulate the way menus are assigned, it may appear desirable that the menus granted to individuals be always as large as possible.

LAR. *Menus as large as possible capable of generating an allocation rule*.
There is no menu M' such that: (i) M' can generate an allocation rule; (ii) for all $P_I^B \in L$ and $i \in I$, $M_i(P_I^B) \subseteq M_i'(P_I^B)$; and (iii) for some $P_I^B \in L$ and $i \in I$, $M_i(P_I^B) \subset M_i'(P_I^B)$.

LAR is motivated by the idea that it is more desirable to be able to choose from a large than from a small menu. Condition (i) in LAR imposes a constraint on how large menus can be: menus cannot be enlarged to the point of being incapable of generating allocation rules.

Remark 2.11. For $n \geq 2 \leq m$, no menu M satisfying SPR and LAR can generate an allocation rule in which every individual receives an object when there are enough objects.

Suppose that M satisfies the required conditions. With $I = \{i, j\}$ and $B = \{x, y\}$, let $P_I^B \in L$ and $Q_I^B \in L$ satisfy ${}^1P_i^B = {}^1Q_j^B = x$ and ${}^1Q_i^B = {}^1P_j^B = y$. In $f(P_I^B)$, each individual must be assigned some object, so $M_i(P_I^B) \neq \varnothing \neq M_j(P_I^B)$. If $M_i(P_I^B) = \{y\}$, then, for M to generate an allocation rule, $M_j(P_I^B) = \{x\}$. In this case, LAR does not hold because M' that differs from M only in that $M_i'(P_I^B) = M_j'(P_I^B) = B$ can generate an allocation rule. If $M_i(P_I^B) = \{x\}$ then M' that differs from M only in that $M_i'(P_I^B) = B$ can also generate an allocation rule. As a result, $M_i(P_I^B) = B$. An analogous reasoning proves that $M_j(Q_I^B) = B$. By SPR, $M_i(P_I^B) = B$ implies $M_i(Q_i^B, P_j^B) = B$, so $f_i(Q_i^B, P_j^B) = y$ and, hence, $y \notin M_j(Q_i^B, P_j^B)$. Since each individual must be assigned some object in $f(Q_i^B, P_j^B)$, $M_j(Q_i^B, P_j^B) \neq \varnothing$. Consequently, $M_j(Q_i^B, P_j^B) = \{x\}$. This and $M_j(Q_I^B) = B$ contradict SPR.

3. Results

Lemma 3.1. Let M be a menu that generates allocation rule f and that satisfies ROB and RIN. Then, for each $x \in A$ and each society I, some $i \in I$ dictates over (x, I) in f.

Proof. Choose $x \in A$ and society I. By definition of allocation rule, there must be $i \in I$ such that $f_i(P_I^{\{x\}}) = x$. It will be shown that i dictates over (x, I). Let $P_J^B \in L$ satisfy $i \in J \subseteq I$ and ${}^1P_i^B = x$. To prove that $f_i(P_J^B) = x$, suppose otherwise: $f_i(P_J^B) \neq x$, so $x \notin M_i(P_J^B)$. With $B = \{x, x_1, \ldots, x_r\}$, by ROB, $x \notin M_i(P_J^B)$ implies $x \notin M_i(P_J^{B\setminus\{x_1\}})$, this implies $x \notin M_i(P_J^{B\setminus\{x_1,x_2\}})$, and so on. Consequently, $x \notin M_i(P_J^{B\setminus\{x_1,\ldots,x_r\}}) = M_i(P_J^{\{x\}})$. On the other hand, $f_i(P_I^{\{x\}}) = x$ implies $M_i(P_I^{\{x\}}) = \{x\}$. With $I\setminus J = \{i_1, \ldots, i_t\}$, by RIN, $M_i(P_I^{\{x\}}) = \{x\}$ implies $M_i(P_{I\setminus\{i_1\}}^{\{x\}}) = \{x\}$, this implies $M_i(P_{I\setminus\{i_1,i_2\}}^{\{x\}}) = \{x\}$, and so on. As a result, $\{x\} = M_i(P_{I\setminus\{i_1,\ldots,i_t\}}^{\{x\}}) = M_i(P_J^{\{x\}})$: contradiction.■

Lemma 3.2. Let M be a menu that generates allocation rule f and that satisfies ROB and RIN. If i dictates over (x, I) in f and j dictates over (y, I) in f, then j dictates over $(x, I\setminus\{i\})$ in f.

Proof. Let i dictate over (x, I) and j over (y, I). With $B = \{x, y\}$, let $P_I^B \in L$ satisfy ${}^1P_i^B = y$ and, for all $k \in I\backslash\{i\}$, ${}^1P_k^B = x$. Since i dictates over (x, I), $M_i(P_I^{\{x\}}) = \{x\}$. By ROB, $x \in M_i(P_I^B)$. And, given that M generates f, $f_i(P_I^B) \neq 0$. Since j dictates over (y, I), $M_j(P_I^{\{y\}}) = \{y\}$. By ROB, $y \in M_j(P_I^B)$. And, given that M generates f, $f_j(P_I^B) \neq 0$. It follows from $f_i(P_I^B) \neq 0 \neq f_j(P_I^B)$ and the fact that M generates f that, for all $k \in I\backslash\{i, j\}$, $M_k(P_I^B) = \varnothing$. By definition of M, there is $k \in I$ such that $M_k(P_I^B) = B$. Since, for all $k \in I\backslash\{i, j\}$, $M_k(P_I^B) = \varnothing$, it must be that $k \in \{i, j\}$. Case 1: $k = j$. In this case, $x \in M_j(P_I^B)$ and, by RIN, $x \in M_j(P_{I\backslash\{i\}}^B)$. Given this, by ROB, $M_j(P_{I\backslash\{i\}}^{\{x\}}) = \{x\}$. As M generates f, $f_j(P_{I\backslash\{i\}}^{\{x\}}) = x$. This and Lemma 3.1 imply that j dictates over $(x, I\backslash\{i\})$. Case 2: $k = i$. Now, $M_i(P_I^B) = B$, so $f_i(P_I^B) = y$. But, as shown, $f_j(P_I^B) \neq 0$. Accordingly, $f_i(P_I^B) = y$ implies $f_j(P_I^B) = x$. By the assumption that M generates f, $x \in M_j(P_I^B)$. As a result, $M_j(P_I^B) = B$, which leads to case 1.■

Proposition 3.3. Let M be a menu that generates allocation rule f. If M satisfies ROB and RIN, then f has a hierarchy of diarchies.

Proof. It has to be shown that, for some sequence $(I_1, \ldots, I_r)$ of societies, (i)-(v) in Definition 2.9 hold. The proof is by induction. Suppose: (1) that sets $I_1, \ldots$, and I_s have been already found that satisfy (ii), (iii), (iv), and (v); or (2) that no such set has been already defined. If (1) is the case, define $I^* = I_1 \cup \ldots \cup I_s$. If (2) is the case, define $I^* = \varnothing$ and $s = 0$. In both cases, the proof amounts to finding a set $I_{s+1} \subseteq N\backslash I^*$ that satisfies (iii), (iv) and (v). By Lemma 3.1, for each $x \in A$, some $i \in N\backslash I^*$ dictates over $(x, N\backslash I^*)$. If all the dictators are the same individual i, then define $I_{s+1} = \{i\}$. If i dictates over $(x, N\backslash I^*)$ and $j \neq i$ over $(y, N\backslash I^*)$, then, by Lemma 3.2, there cannot be a third dictator: if k dictates over $(z, N\backslash I^*)$, then, by Lemma 3.2, both j and z must dictate over $(x, N\backslash(I^* \cup \{i\}))$, which cannot be. In view of this, define $I_{s+1} = \{i, j\}$ and $B = \{x \in A$: i dictates over $(x, N\backslash I^*)\}$. To complete the proof, it must be shown that (v) holds. But this just follows from Lemma 3.2.■

Remark 3.4. If f is an allocation rule having a hierarchy of diarchies, then there is a menu M that generates f and satisfies ROB and RIN.

For allocation rule f having a hierarchy of diarchies $(I_1, \ldots, I_r)$ the following menu M generates f and satisfies ROB and RIN. For $P_I^B \in L$, let $(J_1, \ldots, J_t)$ be the restriction of $(I_1, \ldots, I_r)$ to I. If $J_1 = \{i\}$, then $M_i(P_I^B) = B$. If $J_1 = \{i, j\}$ and ${}^1P_i^B \neq {}^1P_j^B$, then $M_i(P_I^B) = M_j(P_I^B) = B$. If $J_1 = \{i, j\}$ and ${}^1P_i^B = {}^1P_j^B$, then, with $k \in J_1$ dictating over $({}^1P_i^B, I)$, $M_k(P_I^B) = B$ and, for $q \in J_1\backslash\{k\}$, $M_q(P_I^B) = \{f_q(P_I^B)\}$. Finally, for all $k \in I\backslash J_1$, $f_k(P_I^B) = 0$ implies $M_k(P_I^B) = \varnothing$ and $f_k(P_I^B) \neq 0$ implies $M_k(P_I^B) = \{f_k(P_I^B)\}$.

Proposition 3.5. Let M be a menu that generates allocation rule f. If M satisfies SPR, ROB and RIN, then f has a hierarchy of dictators.

Proof. By Proposition 3.3, f has a hierarchy of diarchies. It then suffices to show that each set of the hierarchy has only one member. Specifically, it is enough to prove that if i dictates over (x, I) and j over (y, I), then $i = j$. Suppose not: i dictates over (x, I) and $j \neq i$ over (y, I). Let $J = \{i, j\}$ and $B = \{x, y\}$. Therefore, i dictates over (x, J) and $j \neq i$ over (y, J). Consider the members P_J^B and Q_J^B of L such that ${}^1P_i^B = {}^1P_j^B = x$ and ${}^1Q_i^B = {}^1Q_j^B = y$. Since i dictates over (x, J), $f_i(P_J^B) = x$. By the assumption that M generates f, $x \notin M_j(P_J^B)$. Given that j dictates over (y, J), $f_j(Q_J^B) = y$. By the assumption that M generates f, $y \notin M_i(Q_J^B)$. As i dictates over (x, J), for all $R_J^{\{x\}} \in L$, $M_i(R_J^{\{x\}}) = \{x\}$. By ROB, $x \notin M_i(R_J^B)$ would imply $x \notin M_i(R_J^{\{x\}})$. In view of this,

$$\text{for all } R_J^{\{x\}} \in L,\ x \in M_i(R_J^B). \tag{1}$$

In particular, for $R_J^B = Q_J^B$, $x \in M_i(Q_J^B)$ and $y \notin M_i(Q_J^B)$ imply $M_i(Q_J^B) = \{x\}$. By definition of M, for some $k \in \{i, j\}$, $M_k(Q_J^B) = B$. It then follows from $M_i(Q_J^B) = \{x\}$ that $M_j(Q_J^B) = B$. By SPR, it cannot be that $M_i(Q_J^B) \subset M_i(P_i^B, Q_j^B)$. Accordingly, $M_i(P_i^B, Q_j^B) = \varnothing$, $M_i(P_i^B, Q_j^B) = \{y\}$ or $M_i(P_i^B, Q_j^B) = M_i(Q_J^B) = \{x\}$. But, letting $R_J^B = (P_i^B, Q_j^B)$, by (1), $x \in M_i(P_i^B, Q_j^B)$. Consequently, $M_i(P_i^B, Q_j^B) = \{x\}$. By definition of M, for some $k \in \{i, j\}$, $M_k(P_i^B, Q_j^B) = B$. Hence, $M_j(P_i^B, Q_j^B) = B$. To sum up, $x \notin M_j(P_J^B)$ but $M_j(P_i^B, Q_j^B) = \{x, y\}$, contradicting SPR.■

Remark 3.6. If f is an allocation rule having a hierarchy of dictators, then there is a menu M that generates f and satisfies SPR, ROB and RIN.

For allocation rule f having a hierarchy of dictators $(i_1, \dots, i_r)$ the following menu M generates f and satisfies SPR, ROB and RIN. For $P_I^B \in L$, let $(j_1, \dots, j_t)$ be the restriction of $(i_1, \dots, i_r)$ to I. Then $M_{j_1}(P_I^B) = B$ and, for $s \in \{2, \dots, t\}$, $f_{j_s}(P_I^B) = 0$ implies $M_{j_s}(P_I^B) = \varnothing$ and $f_{j_s}(P_I^B) \neq 0$ implies $M_{j_s}(P_I^B) = \{f_{j_s}(P_I^B)\}$.

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