

Allocation with self-exclusion

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Abstract

Theorem 1 in Ehlers and Klaus (2003, p. 549) characterizes allocation rules having a hierarchy of diarchies in terms of axioms of independence, efficiency and monotonicity. This result does not appear to be correct for the case in which receiving no object may be preferable to receiving some object. A strengthening of independence is shown to be sufficient for the characterization to hold.

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1. Introduction

Ehlers and Klaus (2003) consider the problem of allocating a finite set of objects among a finite set of agents assuming that the set of objects to be allocated may change, that each agent receives at most one object and that the allocation depends on the agents' strict preferences defined over the set of objects together with the null object 0. The null object represents receiving no object but it could be viewed alternatively as a public good. For given sets N of n agents and A of m objects, they consider two domains of preference profiles: the domain R_0 in which the null object is always ranked last and the general domain R in which the null object may occupy any position in a preference.

Theorem 1 in Ehlers and Klaus (2003, p. 549) asserts that, when $m > n$, the only allocation rules, defined on either R or R_0 , satisfying independence, efficiency and monotonicity are those having a hierarchy of diarchies, which are called mixed dictator-pairwise-exchange rules by Ehlers and Klaus (2003, p. 548); see also Ergin (2000), Ehlers (2002) and Ehlers, Klaus and Pápai (2003). That an allocation rule has a hierarchy of diarchies means that there is a ranking of subsets of agents, having each subset either one or two members, such that: (i) it can be interpreted that the members of any given subset G in the hierarchy choose before any other agent coming after in the ranking; and (ii) there is also a ranking between the members of any such G , so that $i \in G$ has priority over $j \in G \setminus \{i\}$ for some objects, whereas j has priority over i for the rest of objects.

This characterization does not appear to be correct for the domain R . To see this, consider the allocation rule f defined as follows: with $N = \{1, 2, 3\}$ and $x \in A$, the hierarchy of dictators 1–2–3 determines the allocation except if 2 prefers the null object to x and x is what both 1 and 3 prefer most, in which case 3 receives x and the hierarchy 1–2 determines the rest of the allocation. Whereas f satisfies independence, efficiency and monotonicity, it does not have a hierarchy of diarchies.

Contrary to what Ehlers and Klaus (2003, p. 554) claim, the proofs for the domains R and R_0 are not analogous. Specifically, Lemma 2 on p. 554 does not hold for R_0 since, as defined on p. 547, independence depends on the position of the null object in the preference. Lemma 2 is based on the idea that, when only one object is going to be assigned, the same agent receives that object in every preference profile. This is guaranteed by independence on R_0 , but not on R . In the previous example, suppose the set of objects to be assigned is $\{x\}$. If x is what all agents prefer most then 1 receives x . But if 2 changes his preference and considers the null object the most preferred option

then it is 3 who receives x . This is consistent with independence because, for 1 to still receive x thanks to independence, all the preferences restricted to $\{x, 0\}$ should be preserved, which is not the case in the present example. In sum, the mapping $f^c(1)$ defined on p. 554 is a function when defined on R_0 but may be a correspondence on R .

This note shows that the characterization can be restored by strengthening independence of irrelevant objects. The new independence axiom suggested (see SIN in Section 3) associates the same allocation to two preference profiles P and Q when: (i) P restricted to the set of objects $B \subseteq A$ to be assigned coincides with Q restricted to B ; (ii) for agents receiving the null object 0 under P , the preference for 0 does not diminish when passing from P to Q ; and (iii) for agents receiving some object under P , that object is still more preferred in Q than 0. If (ii) and (iii) hold, the changes in the preference profile caused by moving the null object up or down the preference rankings may be considered irrelevant. By (ii), no agent receiving 0 has increased the preference of some object over 0, so it is reasonable for those agents to still receive 0. And, by (iii), no agent receiving some object has increased the preference of the null object over the received object, so it is reasonable for those agents to receive the same object as before.

2. Definitions

Let $N = \{1, \dots, n\}$ be a non-empty finite set whose $n \geq 1$ elements represent agents and A a set whose $m \geq 1$ elements are names for objects (or anything that can be assigned to an agent). It is assumed that the symbol “0” does not belong to A . The symbol “0” represents the assignment of no object. For non-empty $B \subseteq A$, $B^+ = B \cup \{0\}$.

Definition 2.1. For $B \subseteq A$, an allocation on B is a mapping $\alpha : N \rightarrow B^+$ such that: (i) for all $i \in N$ and $j \in N \setminus \{i\}$, if $\alpha(i) = \alpha(j) \neq 0$ then $i = j$; (ii) if N has more elements than B then, for every $x \in B$, there is $i \in N$ such that $\alpha(i) = x$; and (iii) if N does not have more elements than B then, for every $i \in N$, there is $x \in B$ such that $\alpha(i) = x$.

An allocation on a subset B of the universal set A of objects is a way of assigning objects to agents (if there are more agents than objects) or agents to objects (otherwise), so that no object is assigned to two different agents and no agent is assigned to two different objects. If $\alpha(i) = x \in B$ then the interpretation is that agent i is assigned object x ; if $\alpha(i) = 0$ then the interpretation is that agent i is assigned no object.

Definition 2.2. A preference on A^+ is a sequence (x_1, \dots, x_{n+1}) such that $\{x_1, \dots, x_{n+1}\} = A \cup \{0\}$.

The definition of preference excludes the possibility of indifference: when $s < t$ in preference $p = (x_1, \dots, x_{n+1})$, x_s is strictly preferred to x_t . Denote by L the set of preferences on A^+ . Denote by L^n the set of all preference profiles on A^+ , that is, the set of ways of assigning a preference on A^+ to each agent. For $p = (x_1, \dots, x_{n+1}) \in L$: (i) $x p y$ means that $x \in A^+$ is strictly preferred to $y \in A^+ \setminus \{x\}$ in p ; (ii) the position $\pi(x, p)$ of x in p is r if, and only if, $x = x_r$; and (iii) for $r \in \{1, \dots, n\}$, ${}^r p$ designates the element that occupies position r in p , so ${}^r p = x_r$. For $p \in L$ and $B \subseteq A^+$ with b elements, define the restriction $p|_B$ of p to B to be such that, for all $x \in B$ and $r \in \{1, \dots, b\}$, ${}^r p|_B = x$ if, and only if, $\{y \in B: y p x\}$ has $r - 1$ members. For $P = (P_i)_{i \in N} \in L^n$ and $B \subseteq A$, $P|_B$ abbreviates $(P_i|_B)_{i \in N}$.

Definition 2.3. Denoting by S the set of all non-empty subsets of A and by A^* the set of all allocations on A , an allocation rule is a mapping $f: L^n \times S \rightarrow A^*$ such that, for all $B \in S$ and $P \in L^n$, $f(P, B)$ is an allocation on B .

An allocation rule takes both a preference profile P and a non-empty subset B of the set of objects A as inputs and outputs an allocation on B . Let f be an allocation rule, $P \in L^n$, $Q \in L^n$, $i \in N$ and $\emptyset \neq B \subseteq A$. Then: (i) P_i designates agent i 's preference in profile P ; (ii) (P_i, Q_{-i}) is the member R of L^n such that $R_i = P_i$ and, for all $j \in N \setminus \{i\}$, $R_j = Q_j$; and (iii) $f_i(P, B)$ designates the object that the allocation $f(P, B)$ on B assigns to i .

Definition 2.4. A diarchy₁ in f is a triple (i, j, D_1) , where $i \in N$, $j \in N$ and $\emptyset \neq D_1 \subset A$, such that, for all $P \in L^n$ and $\emptyset \neq B \subseteq A$:

- if ${}^1 P_i|_{B^+} = {}^1 P_j|_{B^+} = 0$ then $f_i(P, B) = f_j(P, B) = 0$;
- if ${}^1 P_i|_{B^+} = {}^1 P_j|_{B^+} = x \in D_1$ then $f_i(P, B) = x$ and, when $j \neq i$, $f_j(P, B) = {}^1 P_j|_{B^+ \setminus \{x\}}$;
- if ${}^1 P_i|_{B^+} = {}^1 P_j|_{B^+} = y \in A \setminus D_1$ then $f_j(P, B) = y$ and, when $i \neq j$, $f_i(P, B) = {}^1 P_i|_{B^+ \setminus \{y\}}$; and
- if $x = {}^1 P_i|_{B^+} \neq {}^1 P_j|_{B^+} = y$ then $f_i(P, B) = x$ and, when $i \neq j$, $f_j(P, B) = y$.

A diarchy₁ in allocation rule f consists of at most two agents i and j together with a non-empty strict subset D_1 of the set A of objects such that: (i) i receives his most preferred object when this object belongs to D_1 , when it is the null object or when it is different from j 's most preferred object, with j receiving the most preferred object that remains available; and (ii) j receives his most preferred object when this object belongs to $A \setminus D_1$, when it is the null object or when it is different from i 's most preferred object, with i

receiving the most preferred object that remains available. For $r \in \{2, \dots, n\}$, a diarchy $_r$ defined next operates just as a diarchy $_1$ but with respect to the set of objects that have not been assigned to the members of the preceding diarchies.

Definition 2.5. For $r \in \{2, \dots, n\}$, $i \in N$ and $j \in N$, the triple (i, j, D_r) , where $i \in N, j \in N$ and $\emptyset \neq D_r \subset A$, is a diarchy $_r$ in allocation rule f if, for every $s \in \{1, \dots, r-1\}$, there is a diarchy $_s (i_s, j_s, D_s)$ in f such that, for all $P \in L^n$ and $\emptyset \neq B \subseteq A$, where $C = B \setminus \{x \in A: \text{for some } k \in \{i_1, j_1, \dots, i_{r-1}, j_{r-1}\}, f_k(P, B) = x\}$:

- if ${}^1P_i|_{C^+} = {}^1P_j|_{C^+} = 0$ then $f_i(P, B) = f_j(P, B) = 0$;
- if ${}^1P_i|_{C^+} = {}^1P_j|_{C^+} = x \in D_r$ then $f_i(P, B) = x$ and, when $j \neq i$, $f_j(P, B) = {}^1P_j|_{C^+ \setminus \{x\}}$;
- if ${}^1P_i|_{C^+} = {}^1P_j|_{C^+} = y \in A \setminus D_r$ then $f_j(P, B) = y$ and, when $i \neq j$, $f_i(P, B) = {}^1P_i|_{C^+ \setminus \{y\}}$; and
- if $x = {}^1P_i|_{C^+} \neq {}^1P_j|_{C^+} = y$ then $f_i(P, B) = x$ and, when $i \neq j$, $f_j(P, B) = y$.

Definition 2.6. Allocation rule f has a hierarchy of diarchies if there are sets $\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_r, j_r\}$ such that: (i) for all $s \in \{1, \dots, r\}$ and $t \in \{1, \dots, r\} \setminus \{s\}$, $\{i_s, j_s\} \cap \{i_t, j_t\} = \emptyset$; (ii) $N = \{i_1, j_1\} \cup \{i_2, j_2\} \cup \dots \cup \{i_r, j_r\}$; and (iii) for every $s \in \{1, \dots, r\}$, there is a non-empty $D_s \subset A$ such that (i_s, j_s, D_s) is a diarchy $_s$ in f .

3. Assumptions

PAR. For all $B \subseteq A$ and $P \in L^n$, there is no allocation α on B such that: (i) for all $i \in N$, not $f_i(P, B) P_i|_{B^+} \alpha(i)$; and (ii) for some $i \in N$, $\alpha(i) P_i|_{B^+} f_i(P, B)$.

PAR is the standard Pareto principle: with respect to $f(P, B)$, no allocation on B makes someone better off without making someone else worse off. PAR allows agents to be self-excluded from the allocation, as no agent receives an object less preferred than 0.

SIN. For all $B \subseteq A$, $P \in L^n$ and $Q \in L^n \setminus \{P\}$, $Q|_B = P|_B$ implies $f(Q, B) = f(P, B)$ provided that, for all $i \in N$: (i) if $f_i(P, B) = 0$ then $\pi(0, Q_i|_{B^+}) \leq \pi(0, P_i|_{B^+})$; and (ii) if $f_i(P, B) \neq 0$ then $\pi(0, Q_i) \geq \pi(f_i(P, B), Q_i)$.

SIN strengthens the independence of irrelevant objects condition in Ehlers and Klaus (2003, p. 547), which asserts that $Q|_{B^+} = P|_{B^+}$ implies $f(Q, B) = f(P, B)$. SIN declares the preference for 0 to be irrelevant if: (i) agents receiving 0 consider 0 at least as preferred as before; and (ii) agents not receiving 0 consider the object received as preferred to 0.

MON. For all $B \subseteq A$, $P \in L^n$, $x \in A \setminus B$ and $i \in N$, not $f_i(P, B) \succ_i f_i(P, B \cup \{x\})$.

MON is equivalent to resource-monotonicity in Ehlers and Klaus (2003, p. 547): adding another object to the set of objects to be allocated makes no agent worse off. Loosely speaking, f allocates the benefits of growth in such a way that the welfare of no agent is reduced.

4. Result

For a given allocation rule f , $i \in N$, $G \subseteq N \setminus \{i\}$ and $x \in A$: (i) if $G \neq \emptyset$ then $d_i(G, x)$ abbreviates “ $f_i(P, B) = x$, for all $B \subseteq A$ and $P \in L^n$ such that, for all $j \in G$, ${}^1P_j|_{B^+} = 0$ and ${}^1P_i|_{B^+} = x$ ”; and (ii) if $G = \emptyset$ then $d_i(G, x)$ abbreviates “ $f_i(P, B) = x$, for all $B \subseteq A$ and $P \in L^n$ such that ${}^1P_i|_{B^+} = x$ ”. Notice that there cannot be different members i and j of $N \setminus G$ such that both $d_i(G, x)$ and $d_j(G, x)$. The expression $d_i(G, x)$ means that i is a dictator for x when 0 is the most preferred available option for all the agents in $N \setminus G$.

Lemma 4.1. If allocation rule f satisfies PAR, SIN and MON then, for every $x \in A$ and $G \subset N$, there is some $i \in N \setminus G$ such that $d_i(G, x)$.

Proof. Choose $x \in A$, $P \in L^n$ and $G \subset N$ such that, for all $j \in N \setminus G$, $x \succ_j 0$ and, if $G \neq \emptyset$, for all $j \in G$, ${}^1P_j = 0$. By PAR, for all $j \in G$, $f_j(P, \{x\}) = 0$ and, for some $i \in N \setminus G$, $f_i(P, \{x\}) = x$. To prove that $d_i(G, x)$, choose $B \subseteq A$ and $Q \in L^n$ such that ${}^1Q_i|_{B^+} = x$ and, for all $j \in G$, ${}^1Q_j|_{B^+} = 0$. It must be shown that $f_i(Q, B) = x$. Since $Q|_{\{x\}} = P|_{\{x\}}$ and conditions (i) and (ii) in SIN hold, by SIN, $f_i(Q, \{x\}) = f_i(P, \{x\}) = x$. Let $B = \{x, x_1, \dots, x_t\}$. By MON, $x = f_i(Q, \{x\}) = f_i(Q, \{x, x_1\}) = \dots = f_i(Q, \{x, x_1, \dots, x_t\}) = f_i(Q, B)$. ■

Lemma 4.2. If allocation rule f satisfies PAR, SIN and MON then, for all $G \subset N$, the set $\{i \in N: d_i(G, x) \text{ for some } x \in A\}$ has one or two members.

Proof. Choose $G \subset N$. If $m \leq 2$ then the result follows from Lemma 4.1 and the fact that, for all $x \in A$, $i \in N$ and $j \in N$, $d_i(G, x)$ and $d_j(G, x)$ imply $i = j$. If $m \geq 3$ then choose $x \in A$, $y \in A \setminus \{x\}$ and $z \in A \setminus \{x, y\}$. By Lemma 4.1, there are $i \in N \setminus G$, $j \in N \setminus G$ and $k \in N \setminus G$ such that $d_i(G, x)$, $d_j(G, y)$ and $d_k(G, z)$. Let $P \in L^n$ satisfy: $P_j = P_k$; ${}^1P_i = {}^3P_j = z$; ${}^2P_i = {}^2P_j = y$; ${}^3P_i = {}^1P_j = x$; and, when $G \neq \emptyset$, for all $r \in G$, ${}^1P_r = 0$. Since $d_i(G, x)$, $f_i(P, \{x\}) = x$; by MON, $f_i(P, \{x, y\}) \neq 0$. Given $d_j(G, y)$, $f_j(P, \{y\}) = y$; by MON, $f_j(P, \{x, y\}) \neq 0$. As a result, by PAR, $f_j(P, \{x, y\}) = x$. By MON, $f_j(P, \{x, y, z\}) = x$. Consequently,

$$f_k(P, \{x, y, z\}) \neq x. \quad (1)$$

As $d_i(G, x)$, $f_i(P, \{x\}) = x$; by MON, $f_i(P, \{x, z\}) \neq 0$. Since $d_k(G, z)$, $f_k(P, \{z\}) = z$; by MON, $f_k(P, \{x, z\}) \neq 0$. By PAR, $f_k(P, \{x, z\}) = x$. By MON, $f_k(P, \{x, y, z\}) = x$, which contradicts (1). Hence, $\{r \in N: d_r(G, x) \text{ for some } x \in A\}$ does not have three members. ■

Lemma 4.3. Let allocation rule f satisfy PAR, SIN and MON. Then, for all $G \subset N$, $i \in \mathbb{M}G$, $j \in \mathbb{M}(G \cup \{i\})$, $x \in A$ and $y \in A \setminus \{x\}$, if $d_i(G, x)$ and $d_j(G, y)$ then $d_j(G \cup \{i\}, x)$.

Proof. Let $G \subset N$, $i \in \mathbb{M}G$ and $j \in \mathbb{M}(G \cup \{i\})$ be as required. Let $P \in L^n$ satisfy: (i) for all $k \in G$, ${}^1P_k = 0$; (ii) ${}^1P_i = y$, ${}^2P_i = x$ and ${}^3P_i = 0$; (iii) ${}^1P_j = x$ and ${}^2P_j = y$; and (iv) for all $k \in \mathbb{M}(G \cup \{i, j\})$, ${}^1P_k = x$. By $d_i(G, x)$ and $d_j(G, y)$, $f_i(P, \{x\}) = x$ and $f_j(P, \{y\}) = y$. By MON, $f_i(P, \{x, y\}) \neq 0$ and $f_j(P, \{x, y\}) \neq 0$. By PAR, $f_i(P, \{x, y\}) = y$ and $f_j(P, \{x, y\}) = x$. Let $Q \in L^n$ differ from P only in that ${}^2Q_i = 0$ and ${}^3Q_i = x$. By SIN, $f_i(P, \{x, y\}) = y$ and $f_j(P, \{x, y\}) = x$ imply $f_i(Q, \{x, y\}) = y$ and $f_j(Q, \{x, y\}) = x$. Accordingly,

$$\text{for all } k \in \mathbb{M}\{i, j\}, f_k(Q, \{x, y\}) = 0. \quad (2)$$

By PAR, for some $r \in \mathbb{M}\{i\}$, $f_r(Q, \{x\}) = x$. If $r \neq j$ then, by MON, $f_r(Q, \{x, y\}) \neq 0$, contradicting (2). Therefore, $f_j(Q, \{x\}) = x$. Let $R \in L^n$ differ from Q only in that ${}^1R_i = 0$ and ${}^2R_i = y$. By SIN, $f_j(Q, \{x\}) = x$ implies $f_j(R, \{x\}) = x$. Given this, by Lemma 4.1, $d_j(G \cup \{i\}, x)$. ■

Lemma 4.4. Let allocation rule f satisfy PAR, SIN and MON. Then, for all $i \in N$, $j \in \mathbb{M}\{i\}$, $P \in L^n$, $B \subseteq A$ and $x \in B$, if ${}^1P_i|_{B^+} = x$, $d_i(\emptyset, x)$ and there is $z \in A$ such that $d_j(\emptyset, z)$ then $f_i(P, B) = x$ and $f_j(P, B) = {}^1P_j|_{B^+ \setminus \{x\}}$.

Proof. Assume $d_i(\emptyset, x)$, $d_j(\emptyset, z)$ and ${}^1P_i|_{B^+} = x$. Let $y = {}^1P_j|_{B^+ \setminus \{x\}}$. Since $d_i(\emptyset, x)$, $f_i(P, B) = x$. If $y = 0$ then $f_j(P, B) = y$ by PAR. If $y \neq 0$ then, by Lemma 4.2, either $d_j(\emptyset, y)$ or $d_i(\emptyset, y)$. If $d_j(\emptyset, y)$ then $f_j(P, B) = y$. If $d_i(\emptyset, y)$ and $N = \{i, j\}$ then, by PAR, $f_j(P, B) = y$. Finally, if $d_i(\emptyset, y)$ and $N \neq \{i, j\}$ then define $Q \in L^n$ as follows: if $y \neq 0$ then Q differs from P only in that Q_i is obtained from P_i by inserting 0 between x and y ; otherwise, $Q = P$. Since there is $z \in A$ such that $d_j(\emptyset, z)$, by Lemma 4.3, $d_j(\{i\}, y)$. Hence, $f_j(Q, \{y\}) = y$. By MON, $f_j(Q, \{x, y\}) \in \{x, y\}$. Since $d_i(\emptyset, x)$, $f_i(Q, \{x, y\}) = x$. Thus, $f_j(Q, \{x, y\}) = y$. As $P|_{\{x, y\}} = Q|_{\{x, y\}}$ and both (i) and (ii) in SIN hold, by SIN, $f_j(P, \{x, y\}) = y$. By successive application of MON, $f_j(P, \{x, y\}) = y$ implies $f_j(P, B) \in \{x, y\}$. Given that $d_i(\emptyset, x)$ implies $f_i(P, B) = x$, it follows that $f_j(P, B) = y$. ■

Lemma 4.5. If allocation rule f satisfies PAR, SIN and MON then there is a diarchy₁ in f .

Proof. By Lemma 4.2, $J = \{i \in N: d_i(\emptyset, x) \text{ for some } x \in A\}$ has one or two members. If, for some $i \in N$, $J = \{i\}$ then, for any $\emptyset \neq D_1 \subset A$, (i, i, D_1) is a diarchy₁. If there are $i \in N$ and $j \in N \setminus \{i\}$ such that $J = \{i, j\}$ then it will be shown that, for some D_1 , (i, j, D_1) is a diarchy₁. Define $D_1 = \{x \in A: d_i(\emptyset, x)\}$. Choose $P \in L^n$ and $B \subseteq A$, with $x = {}^1P_i|_{B^+}$ and $y = {}^1P_j|_{B^+}$. Case 1: $x = y = 0$. By PAR, $f_i(P, B) = f_j(P, B) = 0$. Case 2: $x = 0 \neq y$. By PAR, $f_i(P, B) = x$ and, by Lemma 4.3, $f_j(P, B) = y$. Case 3: $x \neq 0 = y$. By PAR, $f_j(P, B) = y$ and, by Lemma 4.3, $f_i(P, B) = x$. Case 4: $x \neq 0 \neq y$. Case 4a: $x = y$. If $d_i(\emptyset, x)$ then, by Lemma 4.4, $f_i(P, B) = x$ and $f_j(P, B) = {}^1P_j|_{B^+ \setminus \{x\}}$. If $d_j(\emptyset, y)$ then, by Lemma 4.4, $f_j(P, B) = y$ and $f_i(P, B) = {}^1P_i|_{B^+ \setminus \{y\}}$. Case 4b: $x \neq y$. If either $d_i(\emptyset, x)$ or $d_j(\emptyset, y)$ then, by Lemma 4.4, $f_i(P, B) = x$ and $f_j(P, B) = y$. If neither $d_i(\emptyset, x)$ nor $d_j(\emptyset, y)$ then, by Lemma 4.2, $d_i(\emptyset, y)$ and $d_j(\emptyset, x)$. Let $R \in L^n$ differ from P (if it does) only in that ${}^1R_i|_{B^+} = x$, ${}^2R_i|_{B^+} = y$, ${}^3R_i|_{B^+} = 0$, ${}^1R_j|_{B^+} = y$, ${}^2R_j|_{B^+} = x$ and ${}^3R_j|_{B^+} = 0$. Since $d_i(\emptyset, y)$, $f_i(R, \{y\}) = y$. By MON, $f_i(R, \{x, y\}) \neq 0$. Since $d_j(\emptyset, x)$, $f_j(R, \{x\}) = x$. By MON, $f_j(R, \{x, y\}) \neq 0$. By PAR, $f_i(R, \{x, y\}) = x$ and $f_j(R, \{x, y\}) = y$. As $P|_{\{x, y\}} = R|_{\{x, y\}}$ and both (i) and (ii) in SIN hold, by SIN, $f_i(P, \{x, y\}) = x$ and $f_j(P, \{x, y\}) = y$. By MON, $f_i(P, B) = x$ and $f_j(P, B) = y$. ■

Proposition 4.6. Allocation rule f satisfies PAR, SIN and MON if, and only if, f has a hierarchy of diarchies.

Proof. “ \Leftarrow ” It can be easily verified that if f has a hierarchy of diarchies then f satisfies PAR, SIN and MON. “ \Rightarrow ” Taking Lemma 4.5 as the base of an induction argument, choose $r \geq 2$ and assume that,

$$\text{for all } s \in \{1, \dots, r-1\}, \text{ there is a diarchy}_s (i_s, j_s, D_s) \text{ in } f. \quad (3)$$

Let $G = \{i \in N: \text{for some } s \in \{1, \dots, r-1\}, i \text{ is a member of the diarchy}_s\} = \{i_1, j_1, \dots, i_{r-1}, j_{r-1}\}$. The proof amounts to showing that, if $\bigcap G \neq \emptyset$, there is a diarchy _{r} . Assume $\bigcap G \neq \emptyset$. By Lemma 4.2, $I = \{i \in \bigcap G: d_i(G, x) \text{ for some } x \in A\}$ has one or two members. For $P \in L^n$, let $\Delta(P, B) = B \setminus \{x \in A: \text{for some } i \in G, f_i(P, B) = x\}$. For $x \in A$ and $i \in I$, $D_i(G, x)$ stands for “ $f_i(P, B) = x$, for all $B \subseteq A$ and $P \in L^n$ such that ${}^1P_i|_{\Delta(P, B)^+} = x$ ”.

• Step 1: for all $x \in A$ and $i \in I$, $d_i(G, x)$ implies $D_i(G, x)$. Let $x \in B \subseteq A$ and $P \in L^n$ satisfy ${}^1P_i|_{\Delta(P, B)^+} = x$. It must be shown that $f_i(P, B) = x$. If, for all $k \in G$, ${}^1P_k|_{B^+} = 0$ then, since $d_i(G, x)$, $f_i(P, B) = x$. If, for some $k \in G$, ${}^1P_k|_{B^+} \neq 0$ then B has at least two

members. Define $Q \in L^n$ as follows: (i) for all $k \in G$, $Q_k = P_k$ unless $x P_k 0$, in which case Q_k is obtained from P_k by inserting 0 between $f_k(P, B)$ and x ; and (ii) for all $k \in \mathcal{M}G$, $Q_k = P_k$. Since $d_i(G, x)$, $f_i(Q, \{x\}) = x$. By MON, $f_i(Q, B)$ is x or an object in $\{z \in A: \text{for some } k \in G, f_k(P, B) = z\}$ preferred to x according to P_i . By (3), for all $k \in G$, $f_k(Q, B) = f_k(P, B)$. In view of this, $f_i(Q, B) = x$. Since $P|_B = Q|_B$ and both (i) and (ii) in SIN hold, by SIN, $f_i(P, B) = x$.

- Step 2: if, for some $i \in \mathcal{M}G$, $I = \{i\}$ then, for any $\emptyset \neq D \subset A$, (i, i, D) is a diarchy_r. Let $\emptyset \neq B \subseteq A$ and $P \in L^n$ satisfy ${}^1P_i|_{\Delta(P,B)^+} = x$. If $x = 0$ then, by PAR, $f_i(P, B) = x$. If $x \neq 0$ then, by step 1, $f_i(P, B) = x$.

- Step 3: if there are $i \in \mathcal{M}G$ and $j \in \mathcal{M}(G \cup \{i\})$ such that $I = \{i, j\}$ then, with $D = \{x \in A: d_i(G, x)\}$, (i, j, D) is a diarchy_r. Let $x = {}^1P_i|_{\Delta(P,B)^+}$ and $y = {}^1P_j|_{\Delta(P,B)^+}$. Case 1: $x = y$. If $x = 0$ then, by PAR, $f_i(P, B) = f_j(P, B) = 0$. If $x \neq 0$ then either $x \in D$ or $x \in A \setminus D$. Case 1a: $x \in D$. Let $z = {}^2P_j|_{\Delta(P,B)^+}$. Since $d_i(G, x)$, by step 1, $f_i(P, B) = x$. If $z = 0$ then, by PAR, $f_i(P, B) = z$. If $z \neq 0$ then define $Q \in L^n$ as follows: (i) for all $k \in G \cup \{i\}$, $Q_k = P_k$ unless $z P_k 0$, in which case Q_k is obtained from P_k by inserting 0 between $f_k(P, B)$ and z ; and (ii) for all $k \in \mathcal{M}(G \cup \{i\})$, $Q_k = P_k$. If $d_j(G, z)$ then $f_j(Q, \{z\}) = z$. If $d_i(G, z)$ then, by Lemma 4.3, $d_j(G \cup \{i\}, z)$, so $f_j(Q, \{z\}) = z$. It then follows from $f_j(Q, \{z\}) = z$ and MON that $f_j(Q, B)$ is z or an object in $\{v \in A: \text{for some } k \in G \cup \{i\}, f_k(P, B) = v\}$ that is preferred to z in P_j . By (3), for all $k \in G$, $f_k(Q, B) = f_k(P, B)$. Since $d_i(G, x)$, by step 1, $f_i(Q, B) = x$. Therefore, $f_j(Q, B) = z$. As $P|_B = Q|_B$ and both (i) and (ii) in SIN hold, by SIN, $f_j(P, B) = z$. Case 1b: $x \in A \setminus D$. In the proof of case 1a, swap “ i ” for “ j ”.

Case 2: $x \neq y$. Case 2a: $x = 0 \neq y$. By PAR, $f_i(P, B) = x$. Define $Q \in L^n$ as follows: (i) for all $k \in G$, $Q_k = P_k$ unless $y P_k 0$, in which case Q_k is obtained from P_k by inserting 0 between $f_k(P, B)$ and y ; and (ii) for all $k \in \mathcal{M}G$, $Q_k = P_k$. If $d_j(G, y)$ then $f_j(Q, \{y\}) = y$. If $d_i(G, y)$ then, by Lemma 4.3, $d_j(G \cup \{i\}, y)$, so $f_j(Q, \{y\}) = y$. It then follows from $f_j(Q, \{y\}) = y$ and MON that $f_j(Q, B)$ is y or an object in $\{v \in A: \text{for some } k \in G, f_k(P, B) = v\}$ preferred to y in P_j . By (3), for all $k \in G$, $f_k(Q, B) = f_k(P, B)$. By PAR, $f_i(P, B) = x$. As a result, $f_j(Q, B) = y$. Since $P|_B = Q|_B$ and both (i) and (ii) in SIN hold, by SIN, $f_j(P, B) = y$. Case 2b: $x \neq 0 = y$. In the proof of case 2a, swap “ i ” for “ j ” and “ x ” for “ y ”.

Case 2c: $x \neq 0 \neq y$. Define $Q \in L^n$ as follows: (i) for all $k \in G$, $Q_k = P_k$ unless $x P_k 0$ or $y P_k 0$, in which case Q_k is obtained from P_k by inserting 0 between $f_k(P, B)$ and the most preferred object in $\{x, y\}$ according to P_k ; (ii) $Q_i = P_i$ unless $y P_i 0$, in which case Q_i is obtained from P_i by inserting 0 between x and y ; (iii) $Q_j = P_j$ unless $x P_j 0$, in which case Q_j is obtained from P_j by inserting 0 between y and x ; and (iv) for all $k \in \mathcal{M}(G \cup \{i, j\})$,

$Q_k = P_k$. If $d_i(G, x)$ then $f_i(Q, \{x\}) = x$ and if $d_j(G, x)$ then, by Lemma 4.3, $d_i(G \cup \{j\}, x)$, so $f_i(Q, \{x\}) = x$ as well. By MON, $f_i(Q, B)$ is x or an object in $\{v \in A: \text{for some } k \in G, f_k(P, B) = v\}$ that is preferred to x in Q_i . If $d_j(G, y)$ then $f_j(Q, \{y\}) = y$ and if $d_i(G, y)$ then, by Lemma 4.3, $d_j(G \cup \{i\}, y)$, so also $f_j(Q, \{y\}) = y$. By MON, $f_j(Q, B)$ is y or an object in $\{v \in A: \text{for some } k \in G, f_k(P, B) = v\}$ that is preferred to y in Q_j . By (3), for all $k \in G$, $f_k(Q, B) = f_k(P, B)$. Consequently, $f_i(Q, B) = x$ and $f_j(Q, B) = y$. Since $P|_B = Q|_B$ and both (i) and (ii) in SIN hold, by SIN, $f_i(Q, B) = x$ and $f_j(Q, B) = y$. ■

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