Separable preference aggregation

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Abstract
Social welfare functions are assumed to construct the collective preference by considering information concerning the positions that alternatives occupy in the individuals' preferences. Though non-dictatorial social welfare functions operating in this way exist, they attribute to one of the individuals a considerable power to influence the collective preference. A monotonicity requirement is shown to suffice to generate a weak a dictator.

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1. Introduction

A solution of an aggregation problem is separable if the solution can be obtained as a function of the solution of “parts” of the problem. The addition of natural numbers illustrates this property: the sum xyz of, for instance, 123 and 456 is such that $z = 3 + 6$, $y = 2 + 5$ and $x = 1 + 4$. Since most in economic theory concerns the analysis of how organizations, markets and economies work on the basis of the behaviour of individuals, economic analysis must eventually deal with aggregation problems. These problems typically involve the aggregation of some type of elements characterizing individuals. Fundamental among these elements are the individuals’ preferences.

Arrow’s (1963, p. 97) theorem is perhaps the reference result on preference aggregation; for evidence on the impact of this result in economic theory, see Jerry S. Kelly’s social choice bibliography in http://www.maxwell.syr.edu/maxpages/faculty/jskelly/biblioho.htm. The distinctive assumption in Arrow’s theorem is the independence of irrelevant alternatives (IIA) condition. Numerous variations on IIA have been considered; see, by way of illustration, Baigent (1987), Blair and Pollak (1979), Campbell and Kelly (2000).

The IIA condition expresses a separability requirement because it implicitly presumes that preferences defined on any given set A of alternatives are aggregated by merging preferences defined on subsets of A having two members. For instance, if $A = \{x, y, z\}$, f is the rule that aggregates preferences on A and f is assumed to satisfy IIA then three functions f_{xy}, f_{yz} and f_{xz} determine the way f operates: if, for a certain profile of preferences, $f_{\alpha\beta}$ makes α to be more preferred than [indifferent to, less preferred than] β then f makes α to be more preferred than [indifferent to, less preferred than] β. It is therefore as if f delegated the construction of the new preference to $m(m - 1)/2$ functions, where m is the number of members of the set A of alternatives.

But the fact that these functions cannot operate independently creates potential difficulties: by the transitivity of the collective preference, if, for a certain profile, f_{xy} establishes that x is at least as preferred as y and f_{yz} establishes that y is at least as preferred as z then, for that same profile, f_{xz} is forced to declare x at least as preferred as z. So the larger the number of functions involved, the greater the a priori chance that some incompatibility could emerge. Saari (1998, 2001) criticism to IIA is set in this context: if the aim is to produce a transitive preference, how could one consider reasonable to achieve this goal by decentralizing the production process without any coordination of the production activities at all?

This paper considers a variation on the separability approach to preference aggregation consisting of handling m functions $\{f_x\}_{x \in A}$ rather than $m(m-1)/2$ functions $\{f_{xy}\}_{\{x,y\} \in B}$, where $B = \{C \subseteq A$: C has two members$\}$. The interpretation is that f_x determines the position that x occupies in the collective preference using as inputs the positions that x occupies in the individuals' preferences.

This approach is computationally simpler in that f is broken into a smaller number of functions performing partial aggregations. This reduction in the number of subprocesses involved may contribute to make their compatibility easier. It is also a more direct approach, given that the output is absolute (a position in a preference ordering) and not relative (whether an alternative is above or below another one). In any case, the approach could be viewed as another test to measure the extent to which one can expect from a separable procedure the aggregation of preferences in a reasonable way.

The main results of the paper are three, Proposition 3.6, 4.4. and 5.3. All of them are similar to Arrow's and related theorems, in the sense that the aggregation procedure generates collective preferences that are "too similar" to the preference of one of the individuals. There are, nonetheless, two significant differences. On the one hand, when the individual having the power to determine the collective preference is in a position to exercise this power, he cannot only impose his strict preference but also his indifference. This is in contrast to Arrow's theorem in that Arrow's dictator need not be capable of making his indifference hold at the collective level. On the other hand, in Propositions 3.6 and 4.4, the above individual need not always be in a position to exercise his power (he need not be a dictator): for some preference profiles, the individual cannot determine the collective preference between two alternatives. This is also in contrast to Arrow's theorem, in which the dictator exercises his power over all the domain of preference profiles. Though the assumptions in Proposition 3.6 and 4.4 need not guarantee the existence of a dictator, Proposition 5.3 shows how to obtain a weak dictator.

2. Definitions and assumptions

Let $N = \{1, \dots , n\}$ be a non-empty finite set whose $n \geq 2$ members designate individuals, A a finite set whose $m \geq 1$ elements represent alternatives and T the set of complete and transitive binary relations that can be defined on A, where $\beta \in T$ is: (i) complete if, for all $x \in A$ and $y \in A$, $x\beta y$ or $y\beta x$ or both; and (ii) transitive if, for all $x \in A$, $y \in A\backslash\{x\}$ and $z \in$

$A\backslash\{x, y\}$, $x\beta y$ and $y\beta z$ imply $x\beta z$. Members of T are called “preferences” (on A), whereas members on T^n are called “preference profiles”. For preference profile R and $i \in N$, R_i designates the preference of individual i in profile R. Denote by L the set $\{\beta \in T$: for all $x \in A$ and $y \in A\backslash\{x\}$, $x\beta y$ implies $y\beta x\}$ of preferences in which no indifference occurs.

For $\beta \in T$, $x \in A$ and $y \in A\backslash\{x\}$, x covers y in β if $x\beta y$, not($y\beta x$) and, for all $z \in A\backslash\{x, y\}$, $x\beta z$ implies $y\beta z$. For $\beta \in T$ and $x \in A$, define $\pi(x, \beta)$ to be the position of x in β, where: (i) $\pi(x, \beta) = 1$ if, and only if, for all $y \in A\backslash\{x\}$, $y\beta x$; and (ii) for $p \in \{2, \dots, m\}$, $\pi(x, \beta) = p$ if, and only if, there is $y \in A\backslash\{x\}$ such that $\pi(y, \beta) = p - 1$ and x covers y. Observe that positions are ordered from below: the first position corresponds to the least preferred alternatives whereas, when no indifference occurs, position m corresponds to the most preferred alternative. Observe as well that requiring that the collective preference be a member of T introduces the constraint that if some alternative occupies position $p > 1$ then some other alternative must occupy position $p - 1$. Define $M = \{1, \dots, m\}$ to be the set of the possible positions of an alternative in a preference.

Definition 2.1. A social welfare function is a mapping $f : T^n \to T$ associating a preference with each preference profile.

Definition 2.2. Non-empty $G \subseteq N$ is decisive$_{<m}$ in $g : M^n \to M$ for position $p \in M\backslash\{m\}$ if $g(\xi) = p$ for all $\xi \in M^n$ such that: (i) for all $i \in N$, $\xi_i < m$; and (ii) for all $i \in G$, $\xi_i = p$. Non-empty $G \subseteq N$ is decisive$_{<m}$ in $g : M^n \to M$ if, for all $p \in M\backslash\{m\}$, G is decisive$_{<m}$ in g for position p.

Definition 2.3. Non-empty $G \subseteq N$ is decisive$_{<m}$ in social welfare function f if $\pi(x, f(R)) = p$, for all $x \in A$, $p \in M\backslash\{m\}$ and $R \in T^n$ such that, for all $i \in G$, $\pi(x, R_i) = p$.

That a non-empty subset G of individuals is decisive$_{<m}$ means that, for every position p different from the maximum position m and every alternative x, x is in position p in the collective preference whenever all the members of G rank x in position p.

Definition 2.4. In social welfare function f, individual $i \in N$ is: (i) a dictator when, for all $R \in T^n$, $x \in A$ and $y \in A\backslash\{x\}$, if xR_iy and not(yR_ix) then $xf(R)y$ and not($yf(R)x$); (ii) a weak dictator when, for all $R \in T^n$, $x \in A$ and $y \in A\backslash\{x\}$, if xR_iy and not(yR_ix) then $xf(R)y$; (iii) a strong dictator if, for all $R \in T^n$, $x \in A$ and $y \in A\backslash\{x\}$, xR_iy implies $xf(R)y$; and (iv) a strong dictator$_{<m}$ if $\{i\}$ is decisive$_{<m}$ in f.

A1. For every $x \in A$ there is $f_x : M^n \to M$ such that, for all $R \in T^n$, $\pi(x, f(R)) = f_x(\pi(x, R_1), \dots , \pi(x, R_n))$.

A1 is the requirement that f determines the position of alternative $x \in A$ using only information concerning the position of x in each of the individuals' preferences. It can be viewed as a requirement of positional independence: if x occupies the same positions in two preference profiles then x occupies the same position in the collective preferences that f associates with those profiles.

A2. If f satisfies A1 then, for all $x \in A$ and $p \in M$, $f_x(p, \dots , p) = p$.

By A2, the functions f_x that determine in A1 the position of each $x \in A$ satisfy a property of unanimity: if x is in the same position in all the individual preferences then x occupies that same position in the corresponding collective preference.

3. Nearby dictatorship

It is shown in this section that social welfare functions satisfying A1 and A2 attribute to one of the individuals a considerable influence in the determination of the collective preference. Specifically, there is one individual whose preference exactly coincides with the collective preference whenever all the individuals are indifferent between at least two alternatives. The formal expression of this result is Proposition 3.6, with Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 being instrumental results to prove Proposition 3.6.

By Lemma 3.1, each f_x determining the position of alternative x in the collective preference is a choice or selection function, provided the top position m is not involved: for all $\xi \in M^n$ with $m \notin \{\xi_1, \dots , \xi_n\}$, if x occupies position $f_x(\xi)$ in the collective preference then there is some $i \in N$ such that $f_x(\xi) = \xi_i$. Thus, for x to be in position p in the collective preference under the given constraint some individual must place x in position p in his preference.

Lemma 3.1. With $m \geq 3$, if $f : T^n \to T$ satisfies A1 and A2 then, for all $x \in A$ and $\xi \in M^n$, $m \notin \{\xi_1, \dots , \xi_n\}$ implies $f_x(\xi) \in \{\xi_1, \dots , \xi_n\}$.

Proof. Suppose not: for some $x \in A$ and $\xi \in M^n$ with $m \notin \{\xi_1, \dots , \xi_n\}$, $f_x(\xi_1, \dots , \xi_n) = p \notin \{\xi_1, \dots , \xi_n\}$. Choose $y \in A\backslash\{x\}$, $z \in A\backslash\{x, y\}$ and $R \in T^n$ such that, for all $i \in N$, $\pi(x, R_i) =$

ξ_i, $\pi(y, R_i) = m$ and $\pi(z, R_i) = p$. By A1 and A2, $\pi(y, f(R)) = m$. This means that there is no indifference in $f(R)$; that is, for all $v \in A$ and $w \in A\backslash\{v\}$, $\pi(v, f(R)) \neq \pi(w, f(R))$. But, by A1, $\pi(x, f(R)) = p$, whereas, by A1 and A2, $\pi(z, f(R)) = p$: contradiction.■

By Lemma 3.2, if some subset G of individuals is such that f_x assigns x to position one (from below) when all the members of G rank x in that position and the rest of individuals rank x in position $m - 1$ (one position below the top position) then, no matter the way the rest of individuals rank x below position m, f_x assigns x to position one when all the members of G rank x in that position.

Lemma 3.2. With $m \geq 4$, let $f : T^n \rightarrow T$ satisfy A1 and A2. If there are $x \in A$ and non-empty $G \subset N$ such that $f_x(1^G, (m-1)^{-G}) = 1$ then G is decisive$_{<m}$ in f_x for position 1.

Proof. Let $f_x(1^G, (m-1)^{-G}) = 1$. Arguing inductively, assume $f_x(\xi) = 1$, for any given $\xi \in M^n$ such that, for all $i \in G$, $\xi_i = 1$. The aim is to show that $f_x(\zeta) = 1$, where $\zeta \in M^n$ differs from ξ only in that, for exactly one $j \in N\backslash G$ with $\xi_j > 1$, $\zeta_j = \xi_j - 1$ (if, for all $i \in N$, $\zeta_j = 1$ then $f_x(\zeta) = 1$ by A2). Letting ξ and ζ be as indicated, consider any $S \in T^n$ such that, for some $v \in A\backslash\{x\}$ and $t \in A\backslash\{x, v\}$: (i) for all $i \in N$, $\pi(x, S_i) = \xi_i$ and $\pi(t, S_i) = m$; (ii) for all $i \in N$ with $\xi_i = 1$, $\pi(v, S_i) = 2$; and (iii) for all $i \in N$ with $\xi_i \neq 1$, $\pi(v, S_i) = 1$. As $f_x(\xi) = 1$, by A1, $\pi(x, f(S)) = 1$. By Lemma 3.1, $\pi(v, f(S)) \in \{1, 2\}$. By A2, $\pi(t, f(S)) = m$. Hence, for all $r \in A$ and $s \in A\backslash\{r\}$, $\pi(r, f(S)) \neq \pi(s, f(S))$. As a result, $\pi(x, f(S)) = 1$ implies $\pi(v, f(S)) = 2$. Let $V \in T^n$ differ from S only in that, for all $i \in N$, $\pi(x, V_i) = \zeta_i$. That is, V is obtained from S by permuting x in S_j with the alternative w that x covers.

Case 1: $\zeta_j > 1$ (namely, $w \neq v$). By Lemma 3.1, $\pi(x, f(V)) \in \{1, 2\}$ and $\pi(v, f(V)) \in \{1, 2\}$. As, for all $i \in N$, $\pi(v, V_i) = \pi(v, S_i)$, by A1, $\pi(v, f(V)) = \pi(v, f(S)) = 2$. By A2, $\pi(t, f(V)) = m$. Consequently, for all $r \in A$ and $s \in A\backslash\{r\}$, $\pi(r, f(S)) \neq \pi(s, f(S))$. Therefore, $\pi(x, f(V)) = 1$ and, by A1, $f_x(\zeta) = 1$. Case 2: $\zeta_j = 1$ (so $w = v$). With $q \in A\backslash\{x, v, t\}$ such that $\pi(q, S_j) = 3$, let $W \in T^n$ differ from S only in that $\pi(q, W_j) = \pi(v, S_j) = 1$ and $\pi(v, W_j) = \pi(q, S_j) = 3$. By A1, $\pi(x, f(S)) = 1$ implies $\pi(x, f(W)) = 1$ and $\pi(t, f(S)) = m$ implies $\pi(t, f(W)) = m$. Thus, $\pi(v, f(W)) \neq 1$. Let $Y \in T^n$ differ from V only in that $\pi(q, Y_j) = \pi(v, V_j) = 2$ and $\pi(v, Y_j) = \pi(q, V_j) = 3$. By A1, $\pi(v, f(W)) \neq 1$ implies $\pi(v, f(Y)) \neq 1$. Since, by A1, $\pi(t, f(S)) = m$ implies $\pi(t, f(Y)) = m$, it follows that, for some $d \in A$, $\pi(d, f(Y)) = 1$. By Lemma 3.1, $d \in \{x, v\}$. As $\pi(v, f(Y)) \neq 1$, $\pi(x, f(Y)) = 1$. In view of this, by A1, $f_x(\zeta) = 1$.■

Lemma 3.3 extends the decisiveness$_{<m}$ of group G over position 1 to all positions below m, provided no individual ranks the given alternative in the top position m.

Lemma 3.3. With $m \geq 4$, let $f : T^n \to T$ satisfy A1 and A2. If there are $x \in A$ and non-empty $G \subset N$ such that G is decisive$_{<m}$ in f_x for position 1 then G is decisive$_{<m}$ in f_x.

Proof. Let G be decisive$_{<m}$ in f_x for position 1. Choose $p \in M\backslash\{1, m\}$. The proof amounts to showing that G is decisive$_{<m}$ in f_x for position p. To this end, let $\xi \in M^n$ be such that, for all $i \in G$, $\xi_i = p$ and, for all $i \in N\backslash G$, $\xi_i \neq m$. To prove that $f_x(\xi) = p$, suppose otherwise. Choose $R \in T^n$, $y \in A\backslash\{x\}$, $v \in A\backslash\{x, y\}$ and $z \in A\backslash\{x, y, v\}$ satisfying: (i) for all $i \in N$, $\pi(x, R_i) = \xi_i$ and $\pi(v, R_i) = m$; (ii) for all $i \in G$, $\pi(z, R_i) = 1$; (iii) for all $i \in \{j \in N: \xi_j = p\}$, if $p > 2$ then $\pi(y, R_i) = p - 1$ and if $p = 2$ then $\pi(y, R_i) = p + 1$; and (iv) for all $i \in \{j \in N: \xi_j \neq p\}$, $\pi(y, R_i) = p$. By A2, $\pi(v, f(R)) = m$, so there is a unique $r \in A\backslash\{v\}$ satisfying $\pi(r, f(R)) = p$. By Lemma 3.1, $\pi(r, f(R)) = p$ implies $r \in \{x, y\}$. By A1, it follows from the assumption $f_x(\xi) \neq p$ that $\pi(x, f(R)) \neq p$. Therefore, $r = y$ and $\pi(y, f(R)) = p$. Let $S \in T^n$ differ from R in that, for all $i \in \{j \in N\backslash G: \xi_j \neq p\}$, $\pi(x, S_i) = p - 1$ if $p > 2$ and $\pi(x, S_i) = p + 1$ if $p = 2$. By A1, $\pi(y, f(R)) = p$ implies $\pi(y, f(S)) = p$. Finally, let $V \in T^n$ differ from S in that, for all $i \in \{j \in N\backslash G: \xi_j = p\}$, $\pi(x, V_i) = 1$ and $\pi(z, V_i) = p$. By A2, $\pi(v, f(V)) = m$ and, hence, for all $s \in A$ and $t \in A\backslash\{s\}$, $\pi(s, f(V)) \neq \pi(t, f(V))$. By Lemma 3.1, $\pi(z, f(V)) \in \{1, p\}$. By A2, $\pi(y, f(S)) = p$ implies $\pi(y, f(V)) = p$. But, by the assumption that G is decisive$_{<m}$ in f_x for position 1, $\pi(x, f(V)) = 1$: contradiction.■

By Lemma 3.4, if a group G determines the position of a certain alternative x whenever no individual (within or outside the group) ranks x in the top position m then the group determines the position of all alternatives under the same proviso.

Lemma 3.4. With $m \geq 4$, let $f : T^n \to T$ satisfy A1 and A2. If there are $x \in A$ and non-empty $G \subset N$ such that G is decisive$_{<m}$ in f_x then G is decisive$_{<m}$ in f.

Proof. Let G be decisive$_{<m}$ in f_x. Choose $y \in A\backslash\{x\}$. The proof amounts to showing that G is decisive$_{<m}$ in f_y. By Lemmas 3.2 and 3.3, it is enough to show that $f_y(1^G, (m-1)^{-G}) = 1$. To this end, choose $z \in A\backslash\{x, y\}$ and consider any $R \in T^n$ such that: (i) for all $i \in N$, $\pi(z, R_i) = m$; (ii) for all $i \in G$, $\pi(y, R_i) = 1$ and $\pi(x, R_i) = m - 1$; and (iii) for all $i \in N\backslash G$, $\pi(y, R_i) = m - 1$ and $\pi(x, R_i) = 1$. By A2, $\pi(z, f(R)) = m$. Therefore, for all $r \in A$ and $s \in A\backslash\{r\}$, $\pi(r, f(R)) \neq \pi(s, f(R))$. By Lemma 3.1, $\pi(y, f(R)) \in \{1, m - 1\}$. Since G is decisive$_{<m}$ in f_x, $\pi(x, f(R)) = m - 1$. Accordingly, $\pi(y, f(R)) = 1$ and, by A1, $f_y(1^G, (m-1)^{-G}) = 1$.■

By Lemma 3.5, the power of a group with at least two members to determine the position of an alternative when no individual (within or outside the group) ranks that alternative in the top position m is inherited by some non-empty strict subgroup.

Lemma 3.5. With $m \geq 4$, let $f : T^n \to T$ satisfy A1 and A2. If there is $G \subset N$ having at least two members and decisive$_{<m}$ in f then, for some non-empty $J \subset G$, J is decisive$_{<m}$ in f.

Proof. Let G be decisive$_{<m}$ in f, $J \subset G$ be non-empty and $K = G \backslash J$. Choose $x \in A$, $y \in A\backslash\{x\}$, $z \in A\backslash\{x, y\}$ and $v \in A\backslash\{x, y, z\}$. Consider any $R \in T^n$ such that: (i) for all $i \in N$, $\pi(v, R_i) = m$; (ii) for all $i \in N\backslash G$, $\pi(x, R_i) = m - 1$, $\pi(y, R_i) = 2$ and $\pi(z, R_i) = 1$; (iii) for all $i \in J$, $\pi(x, R_i) = 1$, $\pi(y, R_i) = m - 1$ and $\pi(z, R_i) = 2$; and (iv) for all $i \in K$, $\pi(x, R_i) = m - 1$, $\pi(y, R_i) = 1$ and $\pi(z, R_i) = 2$. Since G is decisive$_{<m}$ in f, $\pi(z, f(R)) = 2$, so there is some $w \in A$ such that $\pi(w, f(R)) = 1$. By A2, $\pi(v, f(R)) = m$, so there is only one such w. By Lemma 3.1, $w \in \{x, y\}$. If $\pi(x, f(R)) = 1$ then, by A1, $f_x(1^J, (m - 1)^{-J}) = 1$ and, by Lemmas 3.2, 3.3 and 3.4, J is decisive$_{<m}$ in f. If $\pi(y, f(R)) = 1$ then let $S \in T^n$ differ from R only in that, for all $i \in N\backslash G$, $\pi(x, S_i) = 1$ and $\pi(z, S_i) = m - 1$. By A1, $\pi(y, f(S)) = \pi(y, f(R)) = 1$. By A2, $\pi(v, f(R)) = m$ and, accordingly, $\pi(x, f(S)) \neq 1$. Let $V \in T^n$ differ from S only in that, for all $i \in N\backslash G$, $\pi(y, V_i) = m - 1$ and $\pi(z, V_i) = 2$. By A2, $\pi(v, f(R)) = m$. This means that there is a unique $t \in A$ such that $\pi(t, f(V)) = 1$. By Lemma 3.1, $t \in \{x, y\}$. By A1, $\pi(x, f(S)) \neq 1$ implies $\pi(x, f(V)) \neq 1$. As a result, $\pi(y, f(V)) = 1$. By A1, $f_y(1^K, (m - 1)^{-K}) = 1$. In view of this, by Lemmas 3.2, 3.3 and 3.4, K is decisive$_{<m}$ in f.■

Proposition 3.6. With $m \geq 4$ and $n \geq 2$, if $f : T^n \to T$ satisfies A1 and A2 then some $i \in N$ is a strong dictator$_{<m}$ in f.

Proof. Choose non-empty $J \subset N$, $z \in A$, $y \in A\backslash\{z\}$ and consider any $R \in T^n$ such that: (i) for all $i \in J$, $\pi(z, R_i) = 1$ and $\pi(y, R_i) = m - 1$; and (ii) for all $i \in N\backslash J$, $\pi(z, R_i) = m - 1$ and $\pi(y, R_i) = 1$. By Lemma 3.1, $\pi(z, f(R)) \in \{1, m - 1\}$. If $\pi(z, f(R)) = 1$ then let $G = J$ and $x = z$; if $\pi(z, f(R)) = m - 1$ then let $G = N\backslash J$ and $x = y$. In both cases, by A1, $f_x(1^G, (m - 1)^{-G}) = 1$. By Lemma 3.2, G is decisive$_{<m}$ in f_x for position 1. By Lemma 3.3, G is decisive$_{<m}$ in f_x. By Lemma 3.4, G is decisive$_{<m}$ in f. If G has one member then is a strong dictator$_{<m}$ in f. Otherwise, by successive application of Lemma 3.5, some $i \in G$ is a strong dictator$_{<m}$ in f.■

Let f be a social welfare function that decentralizes preference aggregation in the following terms: for each alternative x, there is a unanimous function f_x aggregating the positions that

x occupies in the individuals' preferences, so that the position f_x determines is the position that x occupies in the collective preference. In this case, by Proposition 3.6, f attributes a unique individual the power to determine the collective preference when no individual has a preference that linearly orders the m alternatives so that all of them consider some alternative indifferent to another one.

This result is, in a sense, stronger than Arrow's theorem but weaker in another sense. It is stronger in that, on the domain in which the dictator dictates, the dictatorship is absolute: the dictator does not only impose his strict preference (as in Arrow's theorem) but also his indifference, thereby becoming a strong dictator. The result is weaker than Arrow's theorem in that the dictator does not dictate on the whole domain T^n, but just on the set of profiles in which the individuals' preferences define, at most, $m - 1$ positions or when the top alternative is the same in every preference. For the particular case in which the social welfare function asks the individuals to provide preferences without indifferences, the domain on which the dictator completely dictates for sure is easy to compute and can be relatively "small".

Remark 3.7. If $f : L^n \to T$ satisfies A1 and A2, the proportion of profiles in which i is a strong dictator is at least $d_{12} = 1 / m^{n-1}$.

By Proposition 3.6, some i is a strong dictator on $D = \{R \in L^n$: for some $x \in A$ and all $j \in N, \pi(x, R_j) = m\}$, in the sense that, for all $R \in D$, $f(R) = R_i$. To compute the cardinality of D, fix $x \in A$. Since there are $(m - 1)!$ members of L in which x occupies position m, there are $[(m - 1)!]^n$ profiles in which all individuals rank x the mth and, therefore, D has $m[(m - 1)!]^n$ elements. Given that L has $(m!)^n$ elements, $1 / m^{n-1}$ is the proportion of profiles in which strong dictatorship is ensured.

Remark 3.8. The proportion of profiles in which some $i \in N$ is a strong dictator is exactly $1 / m^{n-1}$ in the function $f : L^n \to T$ satisfying A1 and A2 such that, for some $i \in N$, all $R \in T^n$ and all $x \in A$: (i) if $\pi(x, R_i) < m$ then $\pi(x, f(R)) = \pi(x, R_i)$; (ii) and if $\pi(x, R_i) = m$ then $\pi(x, f(R)) = min_{j\in N} \{\pi(x, R_j)\}$.

Remark 3.9. With $m = 3$, let f be the social welfare function $f : T^n \to T$ such that, for all $R \in T^n$ and $x \in A$, $\pi(x, f(R)) = min_{i\in N} \{\pi(x, R_i)\}$. Though f satisfies A1 and A2, it has no strong dictator$_{<m}$. This proves that Proposition 3.6 does not hold when $m = 3$.

4. Almost dictatorship

It is worth noticing that Proposition 3.6 does not invoke one of the assumptions in Arrow's theorem: the Pareto principle. According to this principle, for all $R \in T^n$, $x \in A$ and $y \in A \backslash \{x\}$, $\pi(x, f(R)) > \pi(y, f(R))$ if, for all $i \in N$, $\pi(x, R_i) > \pi(y, R_i)$. Contrary to what might be expected, it is shown in this section that adding this assumption to those in Proposition 3.6 does not create a dictator.

A3. If f satisfies A1 then, for all $x \in A$, $\xi \in M^n$ and $\zeta \in M^n$, if, for all $i \in N$, $\xi_i > \zeta_i$ then $f_x(\xi) > f_x(\zeta)$.

A3 is a weak monotonicity condition stating that if all individuals raise the position of an alternative in their preferences then the position of the alternative in the collective preference is also raised.

Lemma 4.1. If A1 holds then A3 implies A2.

Proof. Assume A1. Let $x \in A$. If $f_x(m, \dots, m) = p < m$ then, by A1 and A3, $f_x(p, \dots, p) < p$. With $f_x(p, \dots, p) = q$, again by A1 and A3, $f_x(q, \dots, q) < q$. Since M has a smallest member, this reasoning will eventually lead to a contradiction of A3. Consequently, $f_x(m, \dots, m) = m$. Choose $r \in M \backslash \{m\}$ and, arguing inductively, suppose that, for all $s \in \{m, m - 1, \dots, r - 1\}$, $f_x(s, \dots, s) = s$. To prove that $f_x(r, \dots, r) = r$, let $f_x(r, \dots, r) = t$. If $t > r$, by A3, $f_x(t, \dots, t) > t$. But, by the induction hypothesis, $f_x(t, \dots, t) = t$: contradiction. If $t < r$ then the line of reasoning initially applied to m will also lead to a contradiction.■

A3*. If f satisfies A1 then, for all $x \in A$, $y \in A \backslash \{x\}$, $\xi \in M^n$ and $\zeta \in M^n$, if, for all $i \in N$, $\xi_i > \zeta_i$ then $f_x(\xi) > f_y(\zeta)$.

In order to make a comparison between A3 and the Pareto principle, A3* is a formulation of the Pareto principle in terms of the functions f_x.

Remark 4.2. A3 implies A3*.

Assume A1. Let $\xi \in M^n$ and $\zeta \in M^n$ be such that, for all $i \in N$, $\xi_i > \zeta_i$. With $x \in A$, $y \in A \backslash \{x\}$ and $q = max\{\zeta_1, \dots, \zeta_n\}$, by A1 and A3, $f_x(\xi) > f_x(q, \dots, q)$. By Lemma 4.1, A2 holds. By A2, $f_x(q, \dots, q) = q = f_y(q, \dots, q)$. Since, for all $i \in N$, $\xi_i > \zeta_i$ and $q = max\{\zeta_1, \dots, \zeta_n\}$, $q < m$. By Lemma 3.1, $f_y(\zeta) \in \{\zeta_1, \dots, \zeta_n\}$. Consequently, $f_y(q, \dots, q) = q \geq f_y(\zeta)$.

Remark 4.3. Given A2, A3* implies A3.

Assume A1. Let $\xi \in M^n$ and $\zeta \in M^n$ be such that, for all $i \in N$, $\xi_i > \zeta_i$. With $x \in A$, $y \in A\backslash\{x\}$ and $q = max\{\zeta_1, \dots, \zeta_n\}$, by A1 and A3*, $f_x(\xi) > f_y(q, \dots, q)$. By A2, $f_y(q, \dots, q) = q = f_x(q, \dots, q)$. Since, for all $i \in N$, $\xi_i > \zeta_i$ and $q = max\{\zeta_1, \dots, \zeta_n\}$, $q < m$. By Lemma 3.1, $f_x(\zeta) \in \{\zeta_1, \dots, \zeta_n\}$. Accordingly, $f_x(q, \dots, q) = q \geq f_x(\zeta)$.

Proposition 4.4. With $m \geq 4$ and $n \geq 2$, if $f : T^n \to T$ satisfies A1 and A3 then there exists a unique $i \in N$ such that for all $R \in \{S \in T^n$: there is no $x \in A$ such that $\pi(x, S_i) = m$ and, for some $j \in N\backslash\{i\}$, $\pi(x, S_j) = 1\}$, $f(R) = R_i$.

Proof. By Lemma 4.1, A1 and A3 imply A2. By Proposition 3.6, let $i \in N$ be the strong dictator$_{<m}$. Let $x \in A$ and $\xi \in M^n$ such that $\xi_i = m$ and, for all $j \in N\backslash\{i\}$, $\xi_j > 1$. Consider $\zeta \in M^n$ such that, for all $j \in N$, $\zeta_j = \xi_j - 1$. Since i a strong dictator$_{<m}$, $f_x(\zeta) = \zeta_i = m - 1$. Hence, by A3, $f_x(\xi) > f_x(\zeta)$. That is, $f_x(\zeta) = m$. This proves that,

$$\text{for all } x \in A \text{ and } \xi \in M^n, \text{ if } \xi_i = m \text{ and, for all } j \in N\backslash\{i\}, \xi_j > 1 \text{ then } f_x(\xi) = m. \qquad (1)$$

Choose $\alpha \in M^n$, $y \in A$ and $p \in M\backslash\{m\}$, with $\alpha_i = p$. The proof concludes by showing that $f_y(\alpha) = p$. Suppose otherwise: $f_y(\alpha) = q \neq p$. Choose $z \in A\backslash\{y\}$ and $v \in A\backslash\{y, z\}$. Consider any $\beta \in M^n$ such that: (i) for all $j \in N$, $\pi(y, R_j) = \alpha_j$; (ii) $\pi(z, R_i) = m$ and $\pi(v, R_i) = q$; (iii) for all $j \in N\backslash\{i\}$, $\pi(z, R_j) = \alpha_j + 1$ if $\alpha_j < m$ and $\pi(z, R_j) = \alpha_j - 1$ if $\alpha_j = m$; and (iv) for all $j \in N\backslash\{i\}$, $\pi(v, R_j) = 1$ if $\alpha_j > 1$ and $\pi(v, R_j) = 2$ if $\alpha_j = 1$. By (1), $\pi(y, f(R)) = m$. Thus, for all $w \in A$ and $t \in A\backslash\{w\}$, $\pi(w, f(R)) \neq \pi(t, f(R))$. Since $f_y(\alpha) = q$, by A1, $\pi(y, f(R)) = q$. And since i a strong dictator$_{<m}$, $\pi(v, f(R)) = q$: contradiction.■

Proposition 4.4 asserts that a social welfare functions that resorts to a family of weakly monotonic functions $\{f_x\}_{x \in A}$ to determine the position of each alternative in the collective preference, must attribute to exactly one individual i the power to establish the collective preference except in the case in which i is not indifferent between two alternatives and some $j \neq i$ regards i's most preferred alternative as his less preferred alternative. This is a sort of veto power acting when i tries to "impose" his most preferred alternative in absence of indifference: if $\pi(x, R_i) = m$, with i being the strong dictator on the set D from

Proposition 4.4, then members of $N\backslash\{i\}$ can avoid the result $f(R) = R_i$ only in case that, for some j, $\pi(x, R_j) = 1$.

Remark 4.5. The social welfare function f defined as follows satisfies A1 and A3 but does not have a dictator (nor, a fortiori, a strong dictator). Choose $i \in N$. Then, for all $R \in T^n$: (i) if there is $x \in A$ such that $\pi(x, R_i) = m$ and, for some $j \in N\backslash\{i\}$, $\pi(x, R_j) = 1$ then $\pi(x, f(R)) = 1$ and, for all $y \in A\backslash\{x\}$, $\pi(y, f(R)) = \pi(y, R_i)$; (ii) otherwise, $f(R) = R_i$.

By Remarks 4.2 and 4.5, the addition of the Pareto principle A3* to the assumptions in Proposition 3.6 does not lead to the existence of a dictator. In contrast, Proposition 4.4 shows that some individual obtains a decisive power on a large domain of preference profiles that is greater than a dictator's power.

Remark 4.6. If $f : L^n \rightarrow T$ satisfies A1 and A3, the proportion of profiles in which the individual i in Proposition 4.4 need not be a strong dictator is, at most, $d_{13} = [(m-1)^{n-1} - 1] / [(m-2)m^{n-1}]$.

With $D = \{R \in T^n$: there is no $x \in A$ such that $\pi(x, R_i) = m$ and, for some $j \in N\backslash\{i\}$, $\pi(x, R_j) = 1\}$, to compute the cardinality $|E|$ of $E = T^n\backslash D$, choose $x \in A$. For $k \in \{1, \ldots, n-1\}$, let $E_k = \{R \in L^n$: $\pi(x, R_i) = m$ and $\{j \in N\backslash\{i\}$: $\pi(x, R_j) = 1\}$ has k elements$\}$. Letting $r = m - 1$, $|E_1| = (r!)^2\ (m! - r!)^{n-2} = r^{n-2}\ (r!)^n$. Similarly, $|E_2| = (r!)^3\ (m! - r!)^{n-3} = r^{n-3}\ (r!)^n$. In general, for $k \in \{1, \ldots, n-1\}$, $|E_k| = r^{n-k-1}\ (r)!^n$. Therefore, with $c = |E_1| + |E_2| + \ldots + |E_{n-1}|$, $c = (1 + r + r^2 + \ldots + r^{n-2})\ (r)!^n$. Recalling that $a + a^2 + a^3 + \ldots + a^t = (a^{t-1} - a) / (a-1)$, $c = [1 + (r^{n-1} - r) / (r-1)]\ (r)!^n$. This is the cardinality of all the members of E in which $\pi(x, R_i) = m$. As there are m members in A, it follows that $|E| = mc$. Since L^n has $(m!)^n$ elements, $d_{13} = mc / (m!)^n = [(r!)^{n-1} - 1] / (r-1)m^{n-1}$.

m	4	5	6	7	5	10	10	10	100	100
n	4	5	6	7	10	5	10	100	10	100
d_{13}	.203	.136	.100	.079	.044	.082	.0484	3.6×10^{-6}	9.3×10^{-3}	3.7×10^{-3}

The table above provides the value of d_{13} for some values of m and n: the i in Proposition 4.4 need not be a strong dictator on, at most, 20'3% of all profiles when $m = n = 4$; 13'6% when $m = n = 5$; 10% when $m = n = 6$; 4.84% when $m = n = 10$; .0003% when $m = 10$ and $n = 100$; .93% when $m = 100$ and $n = 10$; and so on.

Remark 4.7. The proportion of profiles in which the $i \in N$ in Proposition 4.4 is not a strong dictator is exactly d_{13} in the function $f : L^n \to T$ satisfying A1 and A3 such that, for some $i \in N$ and all $R \in T^n$: (i) if there is no $x \in A$ such that $\pi(x, R_i) = m$ then $f(R) = R_i$; and (ii) otherwise, $\pi(x, f(R)) = min_{j \in N} \{\pi(x, R_j)\}$ and, for all $y \in A \backslash \{x\}$, $\pi(y, f(R)) = \pi(y, R_i)$.

Remark 4.8. With $m = 3$, let f be the social welfare function $f : T^n \to T$ such that, for all $R \in T^n$ and $x \in A$, $\pi(x, f(R)) = min_{i \in N} \{\pi(x, R_i)\}$. Though f satisfies A1 and A3, it has no strong dictator$_{<m}$. This proves that Proposition 4.4 does not hold when $m = 3$.

5. Weak dictatorship

It is shown in this section that the powerful individual in Proposition 4.4 becomes a weak dictator by adding a second monotonicity property, A4 next (which is reminiscent of Mas-Colell and Sonnenschein's (1972) positive responsiveness).

A4. If f satisfies A1 then, for all $x \in A$, $\xi \in M^n$ and $\zeta \in M^n$, if, for all $i \in N$, $\xi_i \geq \zeta_i$ then $f_x(\xi) \geq f_x(\zeta)$.

By A4, raising the position of an alternative in some of the preferences, with the position in the rest of preferences unaltered, cannot make the alternative occupy a lower position in the collective preference.

A4*. If f satisfies A1 then, for all $x \in A$, $y \in A \backslash \{x\}$, $\xi \in M^n$ and $\zeta \in M^n$, if, for all $i \in N$, $\xi_i \geq \zeta_i$ then $f_x(\xi) \geq f_y(\zeta)$.

A4* (another Paretian condition) is to A4 what A3* is to A3. By A4*, if no individual ranks alternative x below alternative y then x cannot be below y in the corresponding collective preference.

Remark 5.1. A2 and A4 imply A4*.

Assume A1, A2 and A4. Let $\xi \in M^n$ and $\zeta \in M^n$ be such that, for all $i \in N$, $\xi_i \geq \zeta_i$. With $x \in A$, $y \in A \backslash \{x\}$ and $q = max\{\zeta_1, \dots, \zeta_n\}$, by A1 and A4, $f_x(\xi) \geq f_x(q, \dots, q)$. By A2, $f_x(q, \dots, q) = q = f_y(q, \dots, q)$. By A1 and A4, $f_y(q, \dots, q) \geq f_y(\zeta)$.

Remark 5.2. A4* implies A4.

Assume A1 and A4*. Let $\xi \in M^n$ and $\zeta \in M^n$ satisfy, for all $i \in N$, $\xi_i \geq \zeta_i$. With $x \in A$, $y \in A \backslash \{x\}$ and $q = max\{\zeta_1, \dots, \zeta_n\}$, by A1 and A4*, $f_x(\xi) \geq f_y(q, \dots, q)$ and $f_y(q, \dots, q) \geq f_x(\zeta)$.

Proposition 5.3. With $m \geq 4$ and $n \geq 2$, if $f : T^n \to T$ satisfies A1, A3 and A4 then there exists a unique weak dictator in *f*.

Proof. By Proposition 4.4, let $i \in N$ the strong dictator on $D = \{R \in T^n$: there is no $x \in A$ such that $\pi(x, R_i) = m$ and, for some $j \in N \backslash \{i\}$, $\pi(x, R_j) = 1\}$. It therefore suffices to show that i is a weak dictator on $E = \{R \in T^n$: there is $x \in A$ such that $\pi(x, R_i) = m$ and, for some $j \in N \backslash \{i\}$, $\pi(x, R_j) = 1\}$. Choose $R \in E$ and let $x \in A$ be such that $\pi(x, R_i) = m$ and, for some $j \in N \backslash \{i\}$, $\pi(x, R_j) = 1$. Let $S \in T^n$ differ from R only in that $\pi(x, S_i) = m - 1$. Since $S \in D$, $f(S) = S_i$. This means that, for all $y \in A$, $\pi(y, f(S)) = \pi(y, S_i)$. Therefore, by A1 and A4: (i) $\pi(x, f(R)) \in \{m - 1, m\}$; and (ii) for all $y \in A \backslash \{x\}$, $\pi(y, f(R)) \geq \pi(y, f(S))$. Since, for all $y \in A \backslash \{x\}$ and $j \in N$, $\pi(y, S_j) = \pi(y, R_j)$, by A1, for all $y \in A \backslash \{x\}$, $\pi(y, f(R)) = \pi(y, f(S)) = \pi(y, S_i) = \pi(y, R_i)$. Consequently, there is no $v \in A$ and there is no $w \in A \backslash \{v\}$ such that $\pi(v, R_i) > \pi(w, R_i)$ and $\pi(v, f(R)) < \pi(w, f(R))$, so i is a weak dictator on E.■

By Proposition 4.4, the weak dictator in Proposition 5.3 becomes a strong dictator whenever the alternative he ranks at the top is not ranked at the bottom by some individual. With respect to Proposition 4.4, the addition of A4 makes the i in Proposition 4.4 a weak dictator but does not ensure an increase in the set of profiles in which i is a strong dictator.

Remark 5.4. The proportion of profiles in which some $i \in N$ is a strong dictator is exactly d_{134} in the function $f : L^n \to T$ satisfying A1, A3 and A4 such that, for some $i \in N$ and all $R \in T^n$: (i) if there is no $x \in A$ such that $\pi(x, R_i) = m$ then $f(R) = R_i$; and (ii) otherwise, $\pi(x, f(R)) = min_{j \in N} \{\pi(x, R_j)\}$ and, for all $y \in A \backslash \{x\}$, $\pi(y, f(R)) = \pi(y, R_i)$.

References

ARROW, K. (1963). *Social Choice and Individual Values*, 2nd ed. (New York, Wiley).

BAIGENT, N. (1987). “Twitching Weak Dictators”, *Journal of Economics*, **47**, 407–411.

BLAIR, D. and POLLAK, R. (1979). “Collective Rationality and Dictatorship: The Scope of the Arrow Theorem”, *Journal of Economic Theory*, **21**, 186–194.

CAMPBELL, D. and KELLY, J. (2000). “Information and Preference Aggregation”, *Social Choice and Welfare*, **17**, 3–24.

MAS-COLELL, A. and SONNENSCHEIN, H. (1972). “General Possibility Theorems for Group Decisions”, *Review of Economic Studies*, **39**, 185–192.

SAARI, D. (1998). “Connecting and Resolving Sen’s and Arrow’s Theorems”, *Social Choice and Welfare*, **15**, 239–261.

SAARI, D. (2001). *Decisions and Elections: Explaining the Unexpected* (Cambridge, Cambridge University Press).